Inverse Tension Problems

Rectilinear and Chebyshev Distances

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Outline

- Introduction to Inverse Optimization
- Tension Problems on Networks
- Inverse Minimum Cost Tension Problem under L_1 Norm
- Inverse Minimum Cost Tension Problem under L_∞ Norm
- Inverse Maximum Tension Problem under L_1 Norm
- Conclusions and Future Research

Inverse Optimization

Definition 1. Given an optimization problem and a feasible solution to it, the inverse optimization problem is to find a minimal adjustment of the parameters of the problem (costs, capacities,...) such that the given solution becomes optimum.

$$\text{Optimization problem} \Longrightarrow \text{Forward problem}$$

$$\text{Inverse optimization problem} \Longrightarrow \text{Backward problem}$$

Inverse Optimization - Motivation

- *Geographical Sciences:*

  Predicting the transmission time of the seismic waves in order to model earthquake movements

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- *Traffic Equilibrium:*

  Imposing tolls to change the travel costs so that system optimal flow will be equal to the user equilibrium flow

Tension Problems on Networks

Given $G = (N, A)$ a connected digraph

$\theta \in \mathbb{R}^A$ is a tension on graph G with potential $\pi \in \mathbb{R}^N$ such that

$$\theta_{ij} = \pi_j - \pi_i \quad \forall (i, j) \in A \tag{1}$$

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Properties of tensions [Pla (1971), Rockafellar (1984)]:

- For all cycles C, $\sum_{a_{ij} \in C^+} \theta_{ij} - \sum_{a_{ij} \in C^-} \theta_{ij} = 0$.
- Any linear combination of tensions is a tension.
- A tension is orthogonal to any circulation.

Tension Problems on Networks

Minimum cost tension problem (MCT):

$$\min \sum_{a_{ij} \in A} c_{ij}\theta_{ij} \tag{2}$$

subject to

$$t_{ij} \leq \theta_{ij} \leq T_{ij} \quad \forall a_{ij} \in A$$

θ is a tension

where $t_{ij} \in \mathbb{R} \cup \{-\infty\}$ and $T_{ij} \in \mathbb{R} \cup \{+\infty\}$ are lower and upper bounds.

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Maximum tension problem (MaxT): G contains 2 special nodes, s and t, and an arc $a_{st} \in A$ with bounds $(t_{st}, T_{st}) = (-\infty, \infty)$.

$$\max \ \theta_{st} \tag{3}$$

subject to

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Inverse Tensions - Motivation

- Inverse network flows have been thoroughly analyzed

$$\Longrightarrow$$

Can we extend the results to tensions??

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- Can we find a generalization for linear programs with totally unimodular matrices?

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- Can we find a generalization for linear programs with totally unimodular matrices?

- Inverse tensions might have application in many practical problems.

  Example: Project scheduling where the costs and time can be negociated with the customer.

Inverse MCT - Rectilinear Norm

(Cost) inverse minimum cost tension problem (IMCT$_c$):

A feasible tension $\hat{\theta}$ to a MCT is given

$$\Longrightarrow$$

Find $\hat{c}$: $\hat{\theta}$ is the optimum and $\sum_{a_{ij} \in A} w_{ij} |c - \hat{c}|$ is minimum

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(Cost) inverse minimum cost flow problem (IMCF$_c$):

[Ahuja-Orlin (2002)]

inverse min cost flow under unit weight L_1 norm

$$\equiv$$

min cost flow problem in a unit capacity network

Inverse MCT - Rectilinear Norm

Definition 2. A cut ω is called residual with respect to a tension $\hat{\theta}$ if

$$\forall a_{ij} \in \omega^{+} \quad \hat{\theta}_{ij} < T_{ij}$$
$$\forall a_{ij} \in \omega^{-} \quad \hat{\theta}_{ij} > t_{ij}$$

The cost of a cut ω is:

$$\text{cost}(\omega) = \sum_{a_{ij} \in \omega^{+}} c_{ij} - \sum_{a_{ij} \in \omega^{-}} c_{ij} \tag{4}$$

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Theorem 3. *A tension $\hat{\theta}$ is optimal if and only if all the residual cuts in G have nonnegative costs [Rockafellar (1984)].*

Inverse MCT - Rectilinear Norm

Definition 4. We call the residual cuts ω_1 and ω_2 to be arc-disjoint if

$$\omega_1^+ \cap \omega_2^+ = \emptyset \text{ and } \omega_1^- \cap \omega_2^- = \emptyset$$

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Theorem 5. *Let $\Omega^* = \{\omega_1^*, \omega_2^*, \ldots, \omega_K^*\}$ be the minimum cost collection of arc-disjoint residual cuts in G and $Cost(\Omega^*)$ be its cost. Then, $-Cost(\Omega^*)$ is the optimal objective function value for the inverse minimum cost tension problem under unit weight rectilinear norm.*

Inverse MCT - Rectilinear Norm

LP formulation of the inverse MCT under unit weight L_1 norm is

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{a_{ij} \in A} c_{ij}(\pi_j - \pi_i) \\
\text{subject to} \quad & \\
& -1 \leq \pi_j - \pi_i \leq 1 \qquad \text{for} \quad a_{ij} \in K \\
& 0 \leq \pi_j - \pi_i \leq 1 \qquad \text{for} \quad a_{ij} \in L \\
& -1 \leq \pi_j - \pi_i \leq 0 \qquad \text{for} \quad a_{ij} \in U \\
& \pi \gtrless 0
\end{aligned}
$$

where

$$
\begin{aligned}
K &:= \{a_{ij} \in A : t_{ij} < \hat{\theta}_{ij} < T_{ij}\} \\
L &:= \{a_{ij} \in A : \hat{\theta}_{ij} = t_{ij}\} \\
U &:= \{a_{ij} \in A : \hat{\theta}_{ij} = T_{ij}\}
\end{aligned}
$$

Inverse MCT - Chebyshev Norm

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Inverse MCT - Chebyshev Norm

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[Ahuja-Orlin (2002)]

inverse min cost flow under unit weight L_∞ norm

$$\equiv$$

minimum mean cost cycle problem

Inverse MCT - Chebyshev Norm

ω^* is minimum mean residual cut in G w.r.t. $\hat{\theta}$, i.e.,

$$\mu^* = MCost(\omega^*) = cost(\omega^*)/|\omega^*| \text{ is minimum}$$

Inverse MCT - Chebyshev Norm

ω^* is minimum mean residual cut in G w.r.t. $\hat{\theta}$, i.e.,

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Theorem 6. *Let μ^* denote the mean cost of a minimum mean residual cut in G w.r.t. $\hat{\theta}$. Then, the optimal objective function value for the inverse minimum cost tension problem under unit weight L_∞ norm is $max(0, -\mu^*)$.*

Inverse MCT - Chebyshev Norm

Optimal c^* can be defined as follows:

$$c^*_{ij} = \begin{cases} c_{ij} - \mu^* & \text{if } \hat{\theta}_{ij} < T_{ij} \text{ and } c_{ij} - \varphi_{ij} < 0 \\ c_{ij} + \mu^* & \text{if } \hat{\theta}_{ij} > t_{ij} \text{ and } c_{ij} - \varphi_{ij} > 0 \\ c_{ij} & \text{otherwise} \end{cases}$$

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Minimum mean cost residual cut can be found in strongly polynomial time by a Newton type algorithm [Hadjiat-Maurras (1997)].

Inverse MCT - Chebyshev Norm

LP formulation to inverse MCT under L_∞ norm, i.e., to min mean cost residual cut problem:

$$\text{Minimize} \quad \sum_{a_{ij} \in A} c_{ij}(\pi_j - \pi_i) \tag{5}$$

subject to

$$\sum_{a_{ij} \in A} \eta_{ij} = 1$$

$$-\eta_{ij} \le \pi_j - \pi_i \le \eta_{ij} \quad \text{for } a_{ij} \in K$$

$$0 \le \pi_j - \pi_i \le \eta_{ij} \quad \text{for } a_{ij} \in L$$

$$-\eta_{ij} \le \pi_j - \pi_i \le 0 \quad \text{for } a_{ij} \in U$$

$$\eta \ge 0 \quad \pi \gtrless 0$$

Inverse MCT - Chebyshev Norm

[Radzik (1993)]:

Minimum maximum arc cost problem is dual to max mean weight cut problem

Inverse MCT - Chebyshev Norm

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Minimum maximum arc cost problem:

Find a flow f on G satisfying the demands on nodes while minimizing the maximum arc cost i.e., minimizing $\max_{a_{ij} \in A} c_{ij} f_{ij}$.

Inverse MCT - Chebyshev Norm

Dual of LP (5) is a uniform MMAC on $G' = (N, A')$ where

- The demands/supplies on the nodes are

$$\sum_{j \in N} c_{ji} - \sum_{j \in N} c_{ij} = -Cost(\omega(i)) \ \ \forall i \in N$$

Inverse MCT - Chebyshev Norm

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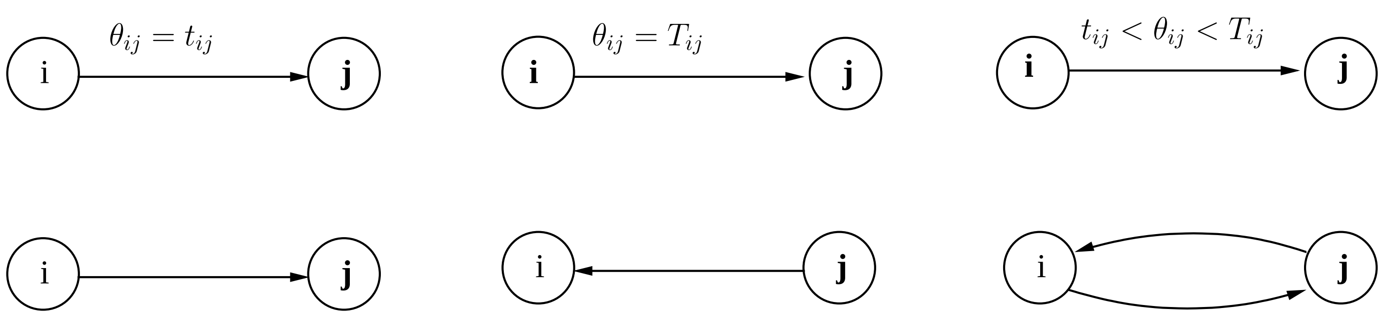
- The demands/supplies on the nodes are

$$\sum_{j \in N} c_{ji} - \sum_{j \in N} c_{ij} = -Cost(\omega(i)) \;\; \forall i \in N$$

- The arc set A' contains



Inverse MaxT - Rectilinear Norm

Inverse maximum tension problem

$$\min \sum_{a_{ij} \in A} w_{ij}(|\hat{T}_{ij} - T_{ij}| + |\hat{t}_{ij} - t_{ij}|) \tag{6}$$

subject to

$$\hat{t}_{ij} \leq \hat{\theta}_{ij} \leq \hat{T}_{ij} \quad \forall a_{ij} \in A$$

$\hat{\theta}_{st}$ is the maximum tension

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Inverse maximum flow problem:

[Yang et al. (1997)]:

inverse maximum flow problem under L_1 norm

$$\equiv$$

maximum flow problem

Inverse MaxT - Rectilinear Norm

Optimality condition [Rockafellar (1984)]

Theorem 7. ***(Maximum Tension Minimum Path Theorem)*** *The maximum in max tension problem is equal to the minimum in min path problem.*

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Theorem 7. ***(Maximum Tension Minimum Path Theorem)*** *The maximum in max tension problem is equal to the minimum in min path problem.*

Property:

If P denotes the minimum path between s and t on graph G and P^+ and P^- are the corresponding sets of forward and backward arcs in P, then $\theta^*_{ij} = T_{ij}$ for all $a_{ij} \in P^+$ and $\theta^*_{ij} = t_{ij}$ for all $a_{ij} \in P^-$ for the maximum tension θ^*.

Inverse MaxT - Rectilinear Norm

Lemma 8. *If the inverse problem has an optimal solution (t^*, T^*) and P^* is the minimum $s - t$ path in network $G = (N, A, t^*, T^*)$, then*

- $T^* \leq T$ and $t^* \geq t$
- $T^*_{ij} = T_{ij}$ and $t^*_{ij} = t_{ij}$ for each arc $a_{ij} \notin P^*$. Moreover, $t^*_{ij} = t_{ij}$ for arcs $a_{ij} \in P^{*+}$ and $T^*_{ij} = T_{ij}$ for arcs $a_{ij} \in P^{*-}$.

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Lemma 8. *If the inverse problem has an optimal solution* (t^*, T^*) *and* P^* *is the minimum* $s - t$ *path in network* $G = (N, A, t^*, T^*)$, *then*

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- $T^*_{ij} = T_{ij}$ and $t^*_{ij} = t_{ij}$ for each arc $a_{ij} \notin P^*$. Moreover, $t^*_{ij} = t_{ij}$ for arcs $a_{ij} \in P^{*+}$ and $T^*_{ij} = T_{ij}$ for arcs $a_{ij} \in P^{*-}$.

Lemma 9. *Inverse maximum tension problem under* L_1 *norm is finding a path* P *from* s *to* t *in* $G = (N, A)$ *such that*

$$\sum_{a_{ij} \in P^+} w_{ij}(T_{ij} - \hat{\theta}_{ij}) + \sum_{a_{ij} \in P^-} w_{ij}(\hat{\theta}_{ij} - t_{ij})$$

is minimum.

Inverse MaxT - Rectilinear Norm

Theorem 10. *The solution to the inverse maximum tension problem under* L_1 *norm with a positive weight function* w *can be found by solving a maximum tension problem in graph* G *with respective upper and lower bounds* $w_{ij}(T_{ij} - \hat{\theta}_{ij})$ *and* $w_{ij}(t_{ij} - \hat{\theta}_{ij})$ *on arcs* $a_{ij} \in A \backslash \{a_{st}\}$.

Conclusions and Future Work

Conclusion:

- Similar results can be proven for inverse tensions as inverse flows.
- Inverse tension problems have "in a way" a **dual** relationship to the inverse flow problems

Future Work:

- Analyzing the capacity inverse minimum cost tension problem

  Flow case: [Güler-Hamacher (2008)]
- Generalization to flows in regular matroids
- Generalization to monotropic optimization
- Exploring the practical applications

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