An application of Network Design with Orientation Constrains

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Outline

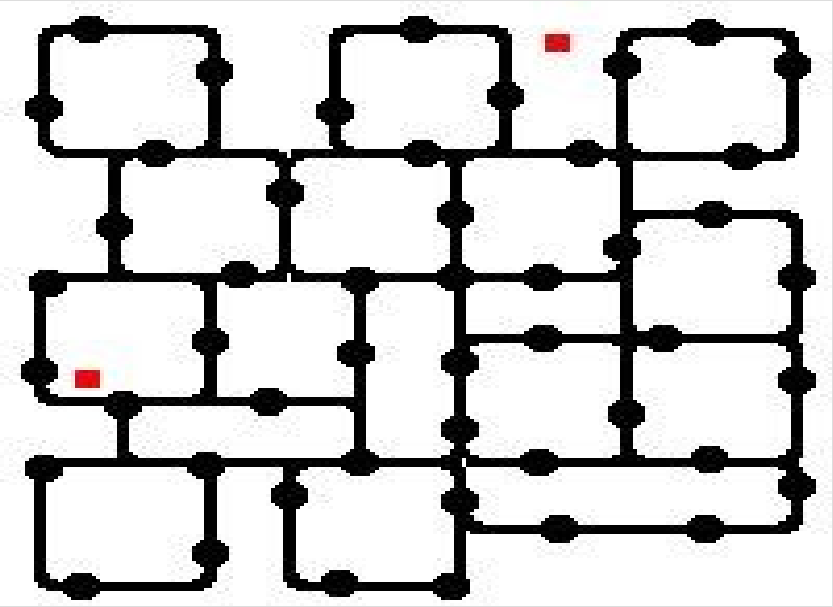
- Motivation
- Problem Description
- Model Description
- Complexity
- Benders Decomposition
- Computational results
- Literature

Motivation 1/2

- This work is motivated by the requirements to design optimal, large scale Personal Rapid Transit (PRT) networks.
- PRT is an innovative type of public transport, composed of fully automated vehicles, running on a dedicated network of one-way guide ways with off-line stations.

Motivation 2/2

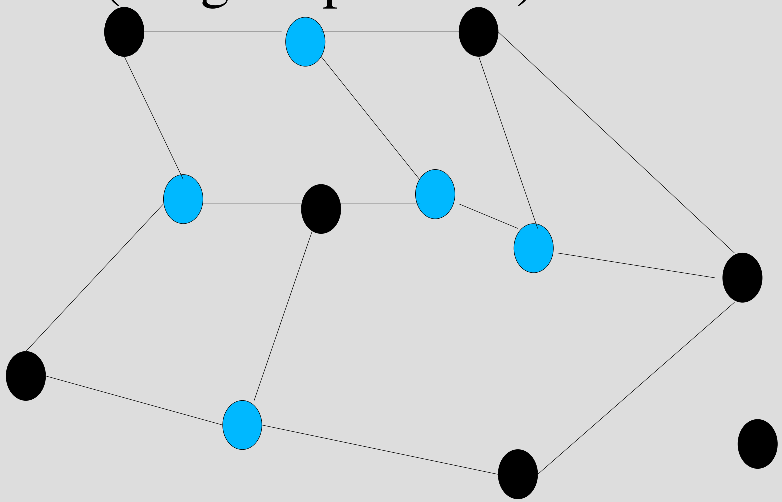
- A crucial issue is to improve the capacity limit of PRT systems through an intelligent design of the network.
- This work deals with the definition of models in order to find an optimized lay-out for a non-trivial PRT network
- Examples of real-world istances:



Problem Description

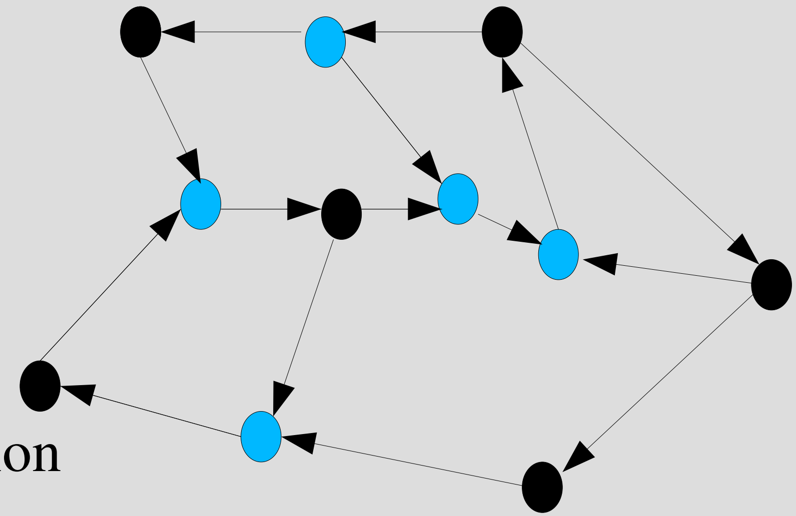
INPUT:

- Undirected Graph G
- OD matrix
- (Edge capacities)



OUTPUT:

- An orientation D of G



● = station

OBJECTIVE:

- Minimize the sum of the Shortest-Path distances between the stations in D (weighted on the OD matrix)

Model Description 1/2

$\overline{A}$

set of the possible arcs arising from orientations of the edges

R

set of origin-destination pairs $r=(s_r, t_r)$ with a demand d_r associated

l_e

lenght of the edge e

$x_{i,j}$

variable equal to one if the edge $\{i, j\}$ is oriented from node i to node j

$y_{i,j}^r$

variable equal to one if the path joining origin s_r to destination t_t uses arc $a\equiv(i, j)\in\overline{A}$

f_i^r

constant equal to 1 if $i=s_r$ and equal to -1 if $i=t_r$, equal to zero otherwise

Model Description 2/2

$\overline{A}$
set of the possible arcs arising from orientations of the edges

R
set of origin-destination pairs $r=(s_r, t_r)$ with a demand d_r associated

l_e
lenght of the edge e

$x_{i,j}$
variable equal to one if the edge $\{i, j\}$ is oriented from node i to node j

$y^r_{i,j}$
variable equal to one if the path joining origin s_r to destination t_t uses arc $a \equiv (i, j) \in \overline{A}$

f^r_i
constant equal to 1 if $i = s_r$ and equal to -1 if $i = t_r$, equal to zero otherwise

$$min \sum_{r \in R} \sum_{a \in \overline{A}} d^r l_a y^r_a$$

$$x_{i,j} + x_{j,i} \leq 1, \quad \forall (i, j) \in E,$$

$$\sum_{a \in \delta^+(i)} y^r_a - \sum_{a \in \delta^-(i)} y^r_a = f^r_i, \quad \forall i \in V, \forall r = (s_r, t_r) \in R,$$

$$y^r_{i,j} \leq x_{i,j}, \quad \forall (i, j) \in \overline{A}, \forall r \in R$$

$$\left(\sum_{r \in R} d^r y^r_a \leq c_a, \quad \forall a \in A \right)$$

$$x_a \in \{0, 1\}, \quad \forall a \in A$$

$$y^r_a \geq 0, \quad \forall a \in A, \forall r \in R$$

Minimize the weighted total routing cost

Orientation constraints for each edge

Flow contraints fo each path

Link constraints between x and y

Capacity constraints NOTE: in this work we will not take into acocount the capacity constraints

Complexity 1/2

Proposition 1:

In case the capacity constraint is imposed testing if the problem has a feasible solution is NP-complete.

Proof :

In case c_e =1 for all e ϵ E, the problem has a solution it and anly if G contains |R| edge-disjointpaths, one from s_r to t_r for r ϵ R. This is well known to be NP-complete.□

Proposition 2:

In case the capacity constraint is not imposed, testing if the problem considered has a feasible solution can be done in linear time.

Proof:

Whitout capacity constraint, the problem has a solution if and only if there exist an orientation of the edges of G such that, for r ϵ R, there exist a direct path from s_r to t_r . Chung, Garey and Tarjan Algorith (1985) proposed an algorithm that test in linear time whether there is an orientation for a mixed graph that preserves strong connectivity and construct such and orientation whenever possible.□

Note that with this algorithm it is possible to obtain an easy-to-compute upper bound.

Complexity 2/2

Theorem :

The problem considered is NP-hard (even in case the capacity constraint is not imposed) .

Proof:

Chvatal and Thomassen (1978) showed that given a graph G = (V, E), finding an orientation of G of diameter 2 is NP-complete.

We can reduce an istance of this problem to our problem (without the capacity constraint) mantaining the same graph G in which $R := \{ (i, j) : i, j \in V, i \neq j \}$, $d_r := 1$ and all edge lenghts $l_e := 1$.

Consider a generic orientation D of G. For each $(i, j) \in R$, if $(i, j) \in E$, one of the two paths will have weight 1, whereas the other one will have weight at least 2; if $(i, j) \notin E$, both paths will have weight at least 2. This proves that the optimal value of our problem is at least $3 |E| + 4 |R \setminus E|$.

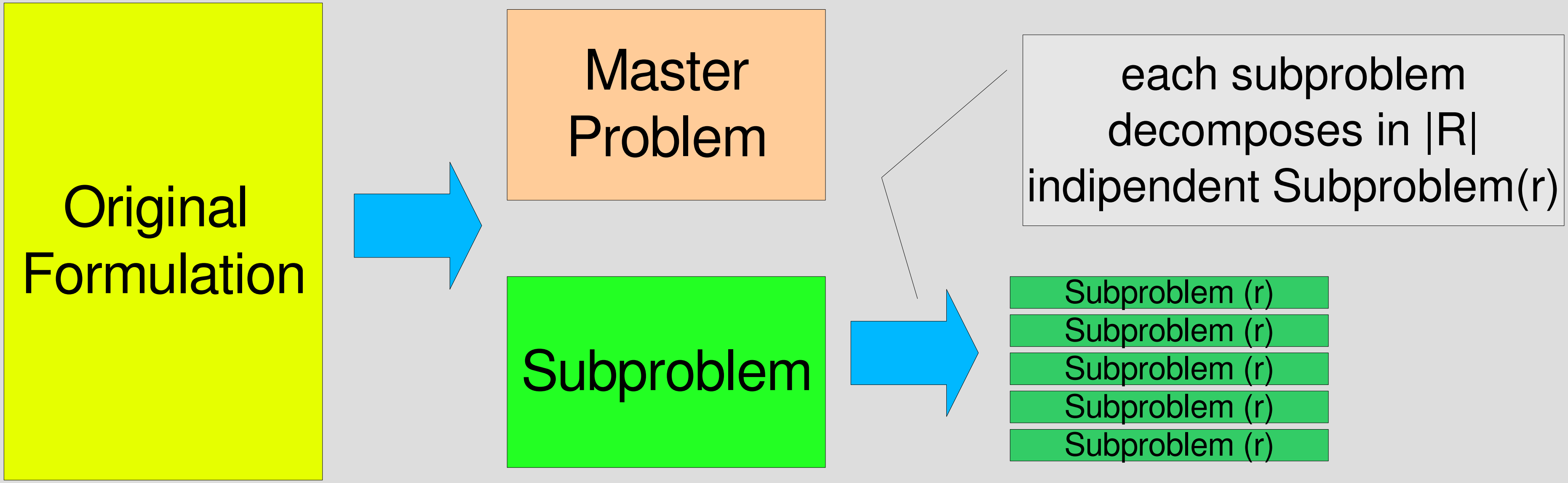
Moreover, the optimal value is exactly $3 |E| + 4 |R \setminus E|$ if and only if there exists an orientation of diameter 2. □

Solution approach: Benders decomposition 1/2

The direct solution of ILP by a general purpose ILP solver quickly becomes impactical as the size of G grows.

In our study we try to raise the size of the solvable instances throught find an efficient way to solve the LP relaxation.

We solve the LP relaxation of the ILP by a Benders decomposition approach, with a Master Problem (MP) with orientation constraints, optimality constraints and feasibility constraints provided by the |R| subproblems.



Solution approach: Benders decomposition 2/2

β^r
variable associated to the contribute of a origin dedstination

E_P^r
set of extreme points of the r-th subproblem

E_R^r
set of extreme rays of the r-th subproblem

MASTER PROBLEM

$$min \sum_{r \in R} \beta^r$$

$$x_{i,j} + x_{j,i} \leq 1, \ \forall (i,j) \in E,$$

$$\beta^r \geq 0, \ \forall r \in R$$

PRIMAL SUBPROBLEM

$$min \sum_{a \in \overline{A}} d^r l_a y_a^r$$

$$\sum_{a \in \delta^+(i)} y_a^r - \sum_{a \in \delta^-(i)} y_a^r = f_i^r, \ \forall i \in V,$$

$$y_{i,j}^r \leq \overline{x_{i,j}}, \ \forall (i,j) \in \overline{A}$$

$$\sum_{r \in R} d^r y_a^r \leq c_a, \ \forall a \in A$$

$$y_a^r \in \{0,1\}, \ \forall a \in A$$

Solution approach: Benders decomposition 2/2

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$$y_{i,j}^r \leq \overline{x_{i,j}}, \ \forall (i,j) \in \overline{A}$$

$$\sum_{r \in R} d^r y_a^r \leq c_a, \ \forall a \in A$$

$$y_a^r \in \{0,1\}, \ \forall a \in A$$

→ DUAL SUBPROBLEM

$$\max \sum_a \overline{x_a} u_a^r - v_{s_r} + v_{t_r}$$

$$-u_a^r - v_i^r + v_j^r \leq d_r l_a, \ \forall (i,j) \in A$$

$$u_a^r \ free, \ \forall a \in A$$

$$v_i^r \geq 0, \ \forall i \in V$$

$\overline{x}$ is the current MP solution

Solution approach: Benders decomposition 2/2

β^r
variable associated to the contribute of a origin dedstination

E_P^r
set of extreme points of the r-th subproblem

E_R^r
set of extreme rays of the r-th subproblem

MASTER PROBLEM

$$\min \sum_{r \in R} \beta^r$$

$$x_{i,j} + x_{j,i} \leq 1, \quad \forall (i,j) \in E,$$

$$\beta^r \geq 0, \quad \forall r \in R$$

$$\sum_{a \in \overline{A}} u_a x_a + \beta_r \geq v_{t_r} - v_{s_r}, \quad \forall r \in R, \forall E_P^r$$

$$\sum_{a \in \overline{A}} u_a x_a \geq v_{t_r} - v_{s_r}, \quad \forall r \in R, \forall E_R^r$$

At each iteration we add up to |R| optimality or feasibility cuts

PRIMAL SUBPROBLEM

$$\min \sum_{a \in \overline{A}} d^r l_a y_a^r$$

$$\sum_{a \in \delta^+(i)} y_a^r - \sum_{a \in \delta^-(i)} y_a^r = f_i^r, \quad \forall i \in V,$$

$$y_{i,j}^r \leq \overline{x_{i,j}}, \quad \forall (i,j) \in \overline{A}$$

$$\sum_{r \in R} d^r y_a^r \leq c_a, \quad \forall a \in A$$

$$y_a^r \in \{0,1\}, \quad \forall a \in A$$

DUAL SUBPROBLEM

$$\max \sum_a \overline{x}_a u_a^r - v_{s_r} + v_{t_r}$$

$$-u_a^r - v_i^r + v_j^r \leq d_r l_a, \quad \forall (i,j) \in A$$

$$u_a^r \ free, \quad \forall a \in A$$

$$v_i^r \geq 0, \quad \forall i \in V$$

$\overline{x}$ is the current MP solution

Pareto optimal cut

Moreover, we add *Pareto-optimal* cuts using the procedure defined by Magnanti and Wong (1981) :

Def (Magnanti and Wong, 1981):

An (optimality) cut $\beta \geq a_1 x + b_1$ *dominates* a cut $\beta \geq a_2 x + b_2$ if $a_1 x + b_1 \geq a_2 x + b_2$ for all x, with a strict inequality for at least one point x. We call a cut *Pareto optimal* if no cut dominates it.

Magnanti and Wong show that, starting from an optimality cut it is possible to obtain a Pareto-optimal cut throght the solution of an auxiliary problem:

Theorem:

Let X be the set of all the feasible points of MP, let x^0 be a point in the interior of X, let $U(x^*)$ and $U(x^0)$ be the set of optimal solutions of the DSP corresponding to x^* and x^0 .

If $u^* \in U(x^*)$ and $u^* \in U(x_0)$ then u^* defines a pareto-optimal cut.

Pareto optimal cut

The new cut can be obtained through the solution of a second LP problem after the solution of the DSP:

DSP(r)

$$max \sum_a \overline{x}_a u_a^r - v_{s_r} + v_{t_r}$$

$$-u_a^r - v_i^r + v_j^r \leq d_r l_a, \quad \forall (i,j) \in A$$

$$u_a^r \; free, \quad \forall a \in A$$

$$v_i^r \geq 0, \quad \forall i \in V$$

DSP(r)-aux

$$max \sum_a x_a^o u_a^r - v_{s_r} + v_{t_r}$$

$$-u_a^r - v_i^r + v_j^r \leq d_r l_a, \quad \forall (i,j) \in A$$

$$\overline{x}_a u_a^r - v_{s_r} + v_{t_r} = z(\overline{x})$$

$$u_a^r \; free, \quad \forall a \in A$$

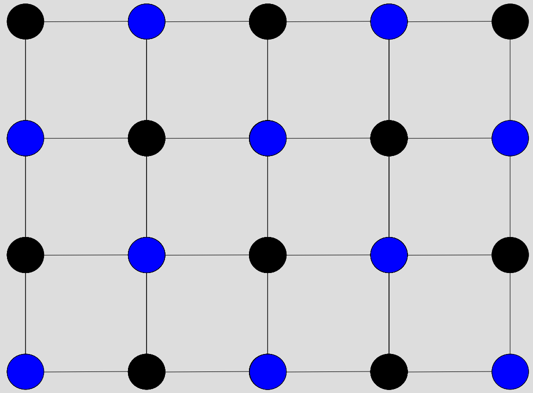
$$v_i^r \geq 0, \quad \forall i \in V$$

the new optimal value must be also an optimal value for DSP(r)

Computational results istances

The istances studied are “ realistc “.

- structure of G: complete grid



- Length of the edges: uniformly distributed in [0.5; 1.5]
- Half of the nodes are stations
- OD matrix: uniformly distributed in [0; 50]

Computational results

Instance	T_Direct	T_Bend	T_Bend P-opt	Val - LP	T_Tarj	Val-Heur
Grid 5x5	1	3	3	11083.05	0	14275.38
Grid 5x6	3	10	6	20417.29	1	25649.4
Grid 6x6	9	31	10	32241.3	2	41978.9
Grid 7x6	20	45	21	47637.86	3	67605.18
Grid 7x7	44	43	89	66257.66	5	90888.58
Grid 8x7	149	459	123	95738.94	10	134738.73
Grid 8x8	294	1653	158	122990.54	19	179941.45
Grid 9x8	577	1566	255	173764.11	27	269328.09
Grid 9x9	989	2829	482	221266.24	42	324867.27
Grid 10x9	2980	Tlim	895	285151.28	52	408750.55
Grid 10x10	Tlim	Tlim	1064	371514.9	83	559278.14
Grid 11x10	Tlim	Tlim	3661	469287.5	139	800488.2

Literature

[1] Anderson, J.E. et al. (1998). Special issue: emerging systems for public transportation. Journal of Advanced Transportation, 32, 1-128.

[2] Chung, F.R.K., Garey, M.R., Tarjan, R.E. (1985). Strongly connected orientations of mixed multigraphs. Networks, 15, 477-484.

[3] Chvatal, V., Thomassen, C. (1978). Distances in orientations of graphs. Journal of Combinatorial Theory Ser. B, 24, 61-75.

[4] Kaspi, M. and Tanchoco, J.M.A. (1990). Optimal ?ow path design of unidirectional AGV systems. International Journal of Production Research, 28,1023-1030.

[5] Langevin, A., Riopel, D., Savard G., Bachmann, R. (2004). A multy-comodity network desing approach fo automated guided vehicle systems. INFOR, 2, 113-123.

[6] Johnson, E.L., Pieroni A. (1983). A linear programming approach to the optimum network orientation problem. Presented at NETFLOW 83: International Workshop on Network Flow Optimization Theory and Practice, Pisa, Italy.

[7] Benders, J.F. (1962). Partitioning procedures for solving mixed variables programming problems. Num. Math, 4, 238-252.

[8] Magnanti, T.L., Mireault, P., Wong, R.T.(1986) Tailoring Benders domposition for uncapacitated network design. Mathematical Programming Study, 26, 112-154