A Note on LP Relaxations for the 1D Cutting Stock Problem with Setup Costs

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Outline of the talk

- The cutting stock problem with setup costs
- A compact mathematical model
- Reformulations by discretization
- Reformulations' lower bound analysis
- The role of damned *big M*
- Some computational results

D cutting stock with setup costs

e 1D Cutting Stock Problem (CSP)

out

- an unlimited number of identical stock items (e.g. steel bars) of given length w;
- a set I of 1D part-types. For each $i \in I$, let $w_i \leq w$ and $d_i > 0$ be respectively th length and the demand of part-type i;

jective

- Satisfy the demand of part types minimizing the number z^* of used stock items

tput

- A set of *cutting patterns* and the number of times each pattern is replicate (*activation levels*).

e Pattern Minimization Problem (PMP)

jective

- Given z^*, minimize the number of different cutting patterns that are used (the number of cutting machine setups)

A mathematical model for PMP

e compact quadratic integer programming formulation (Vanderbe 99) is the natural extension of the assignment formulation for C ntorovich, 1960).

ameters:

w length of stock items

(w_i , d_i) lengths and demands of part-types ($i \in I$)

z^* value of an optimal CSP solution

ary variables:

$$y_j = \begin{cases} 1 & \text{if cutting pattern } j \text{ is used at least once} \\ 0 & \text{otherwise} \end{cases}$$

eger variables:

$_{ij}$ number of items of part-type i cut from a stock item when cutting pattern j is us

$_j$ activation level of cutting pattern j

A mathematical model for PMP

[KAN]

Number of distinct cutting patterns

$$\min \sum_{j=1}^{z^*} y_j$$

mand satisfaction of part type *i* (non linear)

$$\sum_{j=1}^{z^*} z_j x_{ij} = d_i \qquad i \in I$$

feasibility of cutting pattern j

$$\sum_{i \in I} w_i x_{ij} \leq w y_j \qquad j = 1, \ldots, z^*$$

er bound on the number of used stock items

$$\sum_{j=1}^{z^*} z_j \leq z^*$$

activation of cutting pattern j

$$z_j \leq z^* y_j \qquad j = 1, \ldots, z^*$$

- The formulation exhibits some symmetry
- The linearization leads to a linear integer formulation having a weak linear relaxation

ecomposition of integer programs

$$\min \mathbf{c}^{\mathrm{T}}\mathbf{x}$$
$$\mathbf{A}\mathbf{x} \geq \mathbf{b}$$
$$\mathbf{x} \in X$$

where

$$X = P \cap \mathrm{Z}_{+}$$
$P \subseteq \mathrm{R}^{n}$ is a polytope

scretization is an integer analogue to the Dantzig-Wolfe decomposition principle for ear programming.

$$X = \{\mathbf{p}_1,\ldots,\mathbf{p}_q\}$$

$$X = \left\{ \mathbf{x} \in \mathrm{R}^{n} \,\middle|\, \mathbf{x} = \sum_{i=1}^{q} \lambda_i \mathbf{p}_i, \sum_{i=1}^{q} \lambda_i = 1, \boldsymbol{\lambda} \in \{0,1\}^{q} \right\}$$

If $conv(X) \subset P$ the decomposition provides a reformulation whose linear programm relaxation yields a tight lower bound.

Substitution for $\mathbf{x}$ yields an integer program whose linear relaxation is typically sol by column generation

Reformulation [VAN]

$$\min \sum_{j=1}^{z^*} y_j$$

$$\sum_{j=1}^{z^*} z_j x_{ij} = d_i \qquad i \in I$$

$$\sum_{i \in I} w_i x_{ij} \le w y_j \qquad j = 1,\ldots,z^*$$

$$\sum_{j=1}^{z^*} z_j \le z^*$$

$$z_j \le z^* y_j \qquad j = 1,\ldots,z^*$$

Polytope P

[VAN]

$$\min \sum_{k \in K} \sum_{x=1}^{u_k} \lambda_{kx}$$

$$\sum_{k \in K} \sum_{x=1}^{u_k} x a_{ik} \lambda_{kx} = d_i \qquad i \in I$$

$$\sum_{k \in K} \sum_{x=1}^{u_k} x \lambda_{kx} \le z^*$$

$$\lambda_{kx} \in \{0,1\} \qquad k \in K, x = 1,\ldots$$

Parameters

- K set of all the feasible cutting patterns
- a_{ik} number of parts of type i yielded by pattern k
- u_k upper bound to the activation level of pattern k $\quad u_k = \min_{i \in I, a_{ik} > 0} \{z^*, \lfloor d_i / a$

Variables

$\lambda_{kx} = 1$ iff pattern k is applied x times in the solution

Reformulation [VAN]: Pricing Problem

al prices

[VAN]

$$\min \sum_{k\in K}\sum_{x=1}^{u_k}\lambda_{kx}$$

$$\mu: \quad \sum_{k\in K}\sum_{x=1}^{u_k} x a_{ik}\lambda_{kx} = d_i \qquad i\in I$$

$$\sigma: \quad \sum_{k\in K}\sum_{x=1}^{u_k} x\lambda_{kx} \le z^*$$

$$\lambda_{kx}\in\{0,1\} \qquad k\in K, x=1,\ldots,u_k$$

icing Problem

$$\min\left(1 - x\left(\sum_{i\in I}\mu_i q_i - \sigma\right)\right)$$

$$\sum_{i\in I} w_i q_i \le w$$

$$x q_i \le d_i \qquad i\in I$$

$$q_i \ge 0, \text{integer} \qquad i\in I$$

$$x\in\{1,\ldots,x^{\max}\}$$

For fixed x the pricing problem is a boun integer knapsack problem.

The integer non-linear pricing problem can solved by considering a pseudo-polynon number of knapsack problems.

eformulation [GG]

$$\sum_{j=1}^{z^*} y_j$$

$$\sum_{j=1}^{z^*} z_j x_{ij} = d_i \qquad i \in I$$

$$\sum_{i \in I} w_i x_{ij} \le w y_j \qquad j = 1,\dots,z^*$$

$$\sum_{j=1}^{z^*} z_j \le z^*$$

$$z_j \le z^* y_j \qquad j = 1,\dots,z^*$$

Polytope P

[G

$$\min \sum_{k \in K} \lambda_k$$

$$\sum_{k \in K} a_{ik} x_k = d_i \qquad i \in$$

$$\sum_{k \in K} x_k \le z^*$$

$$x_k \le u_k \lambda_k \qquad k \in$$

$$x_k \in \mathrm{N}, \lambda_k \in \{0,1\} \qquad k \in$$

rameters

- K set of all the feasible cutting patterns
- a_{ik} number of parts of type i yielded by pattern k
- u_k upper bound to the activation level of pattern k $\quad u_k = \min_{i \in I, a_{ik} > 0} \{z^*, \lfloor d_i / a_i$

ariables

- x_k activation level of pattern k
- $\lambda_k = 1$ if pattern k is applied at least once

Reformulation [GG]: Pricing Problem

$$\min \sum_{k \in K} \lambda_k \qquad \text{[GG]}$$

l prices

$$\begin{cases} \mu: & \sum_{k \in K} a_{ik} x_k = d_i & i \in I \\ \sigma: & \sum_{k \in K} x_k \le z^* & \\ \gamma: & x_k \le u_k \lambda_k & k \in K \\ & x_k \in \mathrm{N}, \lambda_k \in \{0,1\} & k \in K \end{cases}$$

rimal variables (x_k , λ_k) correspond to dual inequalities

$$\begin{cases} \sum_{i \in I} a_{ik} \mu_i - \sigma - \gamma_k \le 0 \\ \gamma_k u_k \le 1 \end{cases}$$

(μ , σ, γ) is dual feasible if

$$1 - u_k \left(\sum_{i \in I} a_{ik} \mu_i - \sigma \right) \ge 0$$

a profitable column ($\mathbf{a}_k$, u_k) is a cutting pattern satisfying

$$1 - u_k \left(\sum_{i \in I} a_{ik} \mu_i - \sigma \right) < 0$$

[VAN] vs. [GG]

	[VAN]	[GG]
# of constraints	$\|I\| + 1$	$O(2^{\|I\|})$
# of variables	$O(z^* \cdot 2^{\|I\|})$	$O(2^{\|I\|})$
Pricing problem	The **same** non-linear integer program	
Column generation	Pricing can be directly embedded in a column generation scheme	Column-and-row generation scheme is required

On lower bounds by linear relaxation

[VAN^{LP}], [GG^{LP}]:	linear relaxation of [VAN] and [GG]
$z_{VAN}^{LP}(\mathbf{u})$, $z_{GG}^{LP}(\mathbf{u})$:	optimal value of programs [VAN^{LP}] and [GG^{LP}]

Observation

[VAN] and [GG] can be obtained by <u>mere</u> dualization of [KAN] *only if* $u_k = z^*$. In this ca
Lagrangian theory tells us that formulation [GG^{LP}] cannot be stronger than [VAN^{LP}].
both of them use specific upper bounds u_k instead of the trivial value z^*.

Proposition

$z_{VAN}^{LP}(\mathbf{u}) \leq z_{GG}^{LP}(\mathbf{u})$ and the inequality holds strict for some vector $\mathbf{u} \in R^{|K|}$ of upper bound
the activation level of cutting patterns.

<u>*Proof*</u>: Dantzig-Wolfe decomposition applied to [GG] transforms it into [VAN] plus

$$\sum_{x=1}^{u_k} \lambda_{kx} \leq 1 \qquad k \in K \qquad (1)$$

The linear relaxation of [VAN] + (1) is always equivalent to [GG^{LP}] and better than
[VAN^{LP}] when $u_k < z^*$

On lower bounds by linear relaxation

ontinue

act, the convexification of constraints $x_k \le u_k\lambda_k$ in [GG] has no effect on $z_{GG}^{LP}(\mathbf{u})$ sinc

polyhedra

$$P_k = \{(x_k, \lambda_k) \in R^2 | x_k - u_k\lambda_k \le 0, x_k \ge 0, 0 \le \lambda_k \le 1\} \qquad k \in K$$

e integral extreme points (0,0), (0,1), $(u_k, 1)$.

he other hand

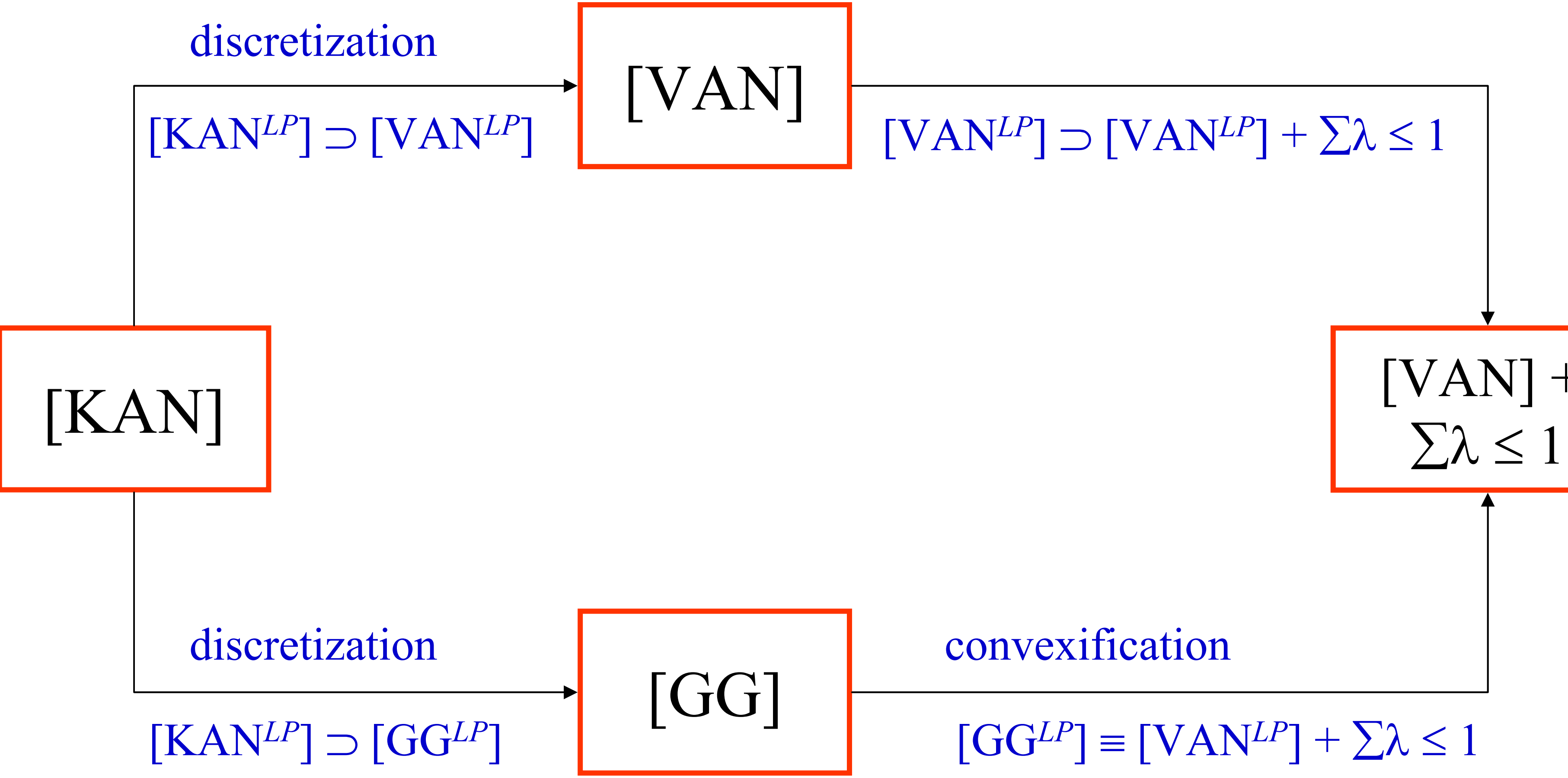
$$z_V^{LP}(\mathbf{u}) = \frac{89}{88} < \frac{228}{225} = z_{GG}^{LP}(\mathbf{u})$$

$= \{1,2\}$, $w = 50$, $(w_1, d_1) = (8,182)$, $(w_2, d_2) = (9,91)$, $\quad u_k = \min_{i \in I, a_{ik} > 0} \{z^*, \lfloor d_i / a_{ik} \rfloor\}$

mark

en $u_k = z^*$ inequalities (1) are redundant in [VAN^{LP}]

he whole decomposition scheme



Computing the upper bound u_k

arts cannot be produced more than required $u_k^V \le \lfloor d_i / a_{ik} \rfloor \quad i \in I, a_{ik} > 0$
Vanderbeck, 1999)

$$u_k^V = \min_{i \in I, a_{ik} > 0} \{z^*, \lfloor d_i / a_{ik} \rfloor\}$$

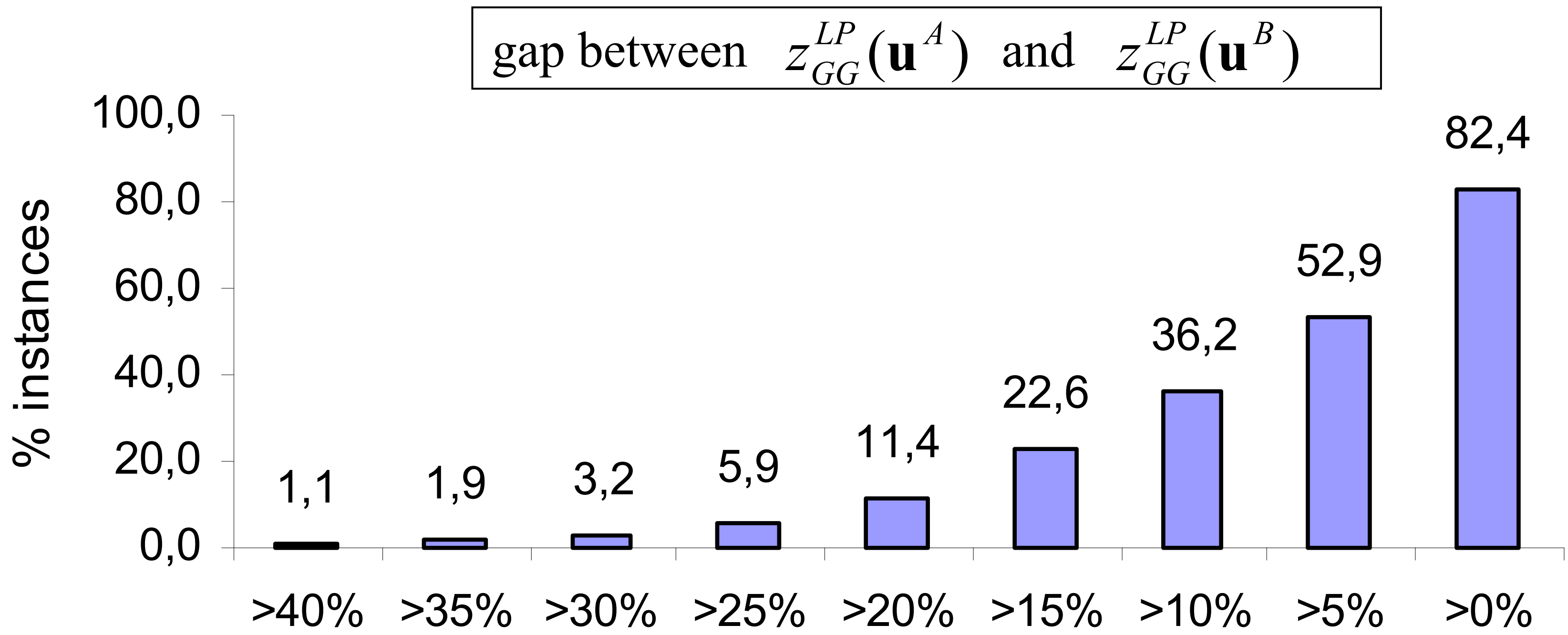
he waste yielded by cutting pattern k cannot greater than the total waste (Alves, 2005) $u_k^A \left(w - \sum_{i \in I} w_i a_{ik}\right) \le z^* w - \sum_{i \in I} w$

$$u_k^A = \min\left\{ \frac{z^* w - \sum_{i \in I} w_i d_i}{w - \sum_{i \in I} w_i a_{ik}}, \min_{i \in I, a_{ik} > 0} \{\lfloor d_i / a_{ik} \rfloor\} \right\}$$

ctivating pattern k at level u_k might be infeasible, because the remaining $(z^* - u_k)$ stoc
ns might not be sufficiently many to cover the demand $\mathbf{d}'$ not yet fulfilled. This is the c
$z'\rceil > z^* - u_k$ where z' is a lower bound to the optimal value of 1-CSP defined on $\mathbf{d}'$. In
way a better bound u_k^B can be obtained.

Computational experience

375 random instances generated by *Cutgen1*, with $|I| = 10$, $w = 1000$, and mean part
e demand $d = 50$.



The linear relaxation lower bound computed by using $\mathbf{u}^B$ is better in 310 cases.

In 43 cases $z_{GG}^{LP}(\mathbf{u}^B)$ improves the bound by more than 20%.

Conclusions

e show that reformulation [GG] is in general better than [VAN], and, at st from the theoretical point of view, it should not be discarded.

e observe that the upper bounds on the pattern activation levels play a ucial role in the implementation of a good and practical exact algorithm.

e propose a method to improve the better available upper bound

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2. Reformulations and lower bounds by linear relaxation

Reformulating [Kan] by discretization, see [6], gives tighter bounds to the 1-PMP. Indeed, different master formulations can be drawn from [Kan], depending on the set of dualized constraints. In [5], the author describes a 1-PMP master formulation [Van] obtained by dualizing (1) and (3), or equivalently,

from discretization of the polyedron defined by (2) and (4)-(8).

An alternative master formulation [GG], very close to that of Gilmore and Gomory for the 1-CSP [2] with the addition of fixed setup costs, derives from discretizing the polyedron defined by (2), (6) and (7).

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