On the Bi-Enhancement of Chordal-Bipartite Probe Graphs

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Outline

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Definitions

- C_k – chordless cycle of size k.
- Chordal graph – a graph with no induced C_k, for $k>3$.
- Chordal-bipartite graph – a bipartite graph that has no induced C_k, for $k>4$.
- Interval graph is the intersection graph of a set of intervals on the real line.

The $\mathscr{C}$-probe problem

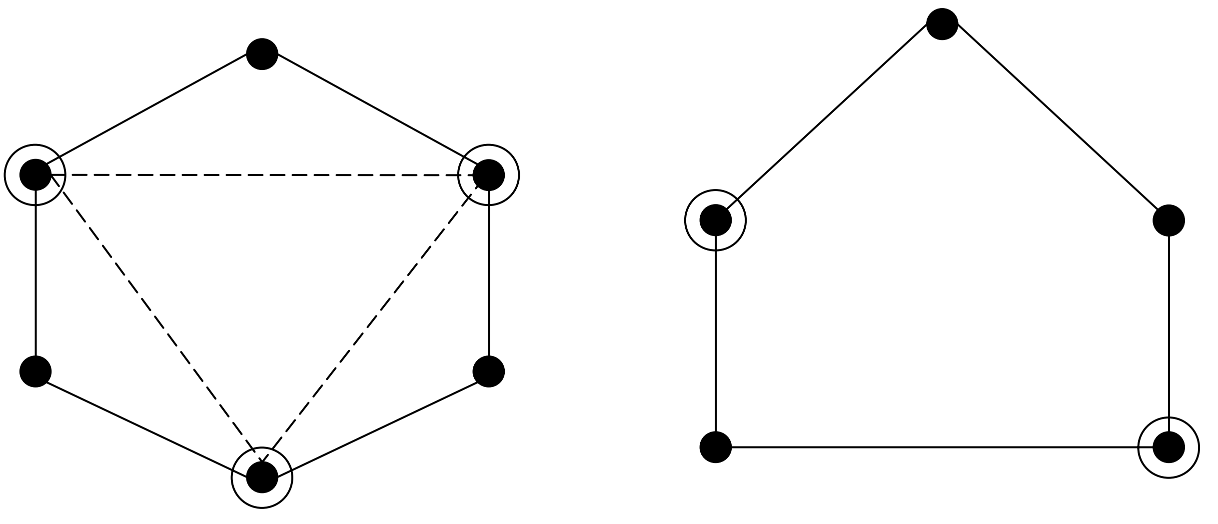
- Let $\mathscr{C}$ be a graph family.
- A graph G=(V,E) is called $\mathscr{C}$-*probe* if
  1. $\exists$ partition V=P(probes) $\cup$ N(non-probes), N is a stable set
  2. $\exists$ completion E' $\subseteq$ {(u,v)|u,v$\in$N}
     such that G'=(V,E$\cup$E') is in $\mathscr{C}$.

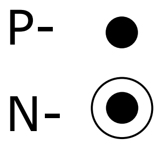
  (Clearly G_P is in $\mathscr{C}$)

- In the *partitioned* version the partition into probes and non-probes is given and fixed.

The $\mathcal{C}$-probe problem - example

- Let $\mathcal{C}$ be the chordal graphs family.



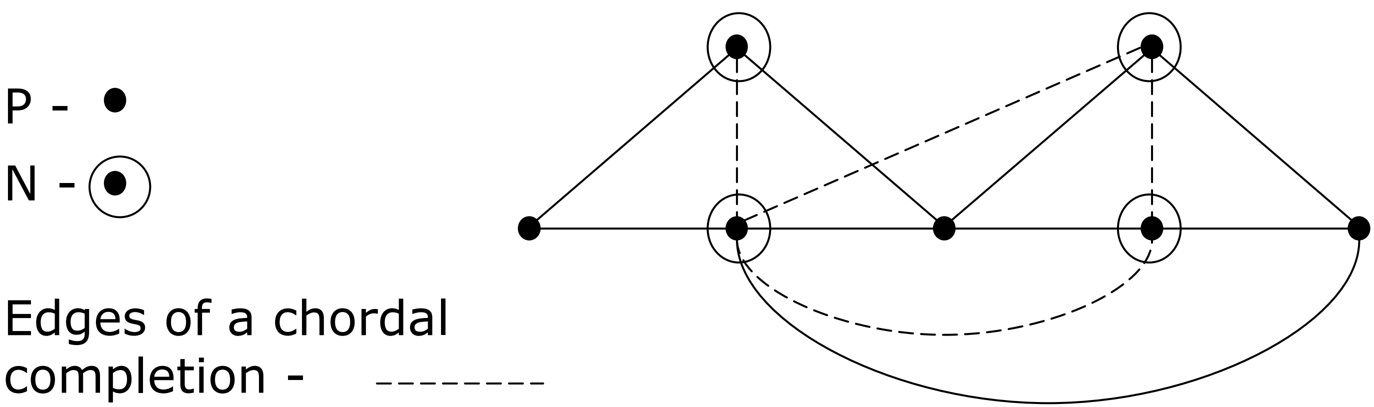


Previous work

- “Probe interval graphs and their application to physical mapping of DNA” [Zhang, 1994]
- “Chordal probe graphs” [Golumbic, Lipshteyn, 2004]
- “Cycle-bicolorable graphs and triangulating chordal probe graphs” [Berry, Golumbic, Lipshteyn, 2007]

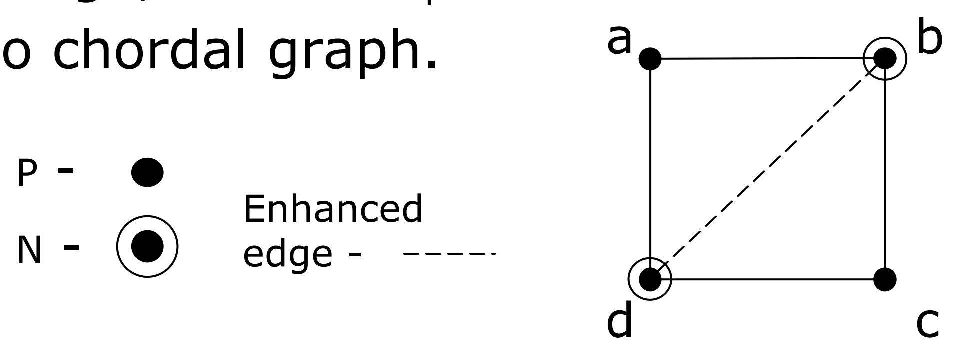
Chordal probe – a necessary condition

- *Lemma 1* – If G is a *chordal probe* graph with respect to the partition $P \cup N$, then probes and non-probes must alternate on every chordless cycle.



Chordal probe - C_4 case

- The C_4 graph is a special case of chordal probe graphs where it has:
  1. Alternating probe/non-probe vertices.
  2. An edge is forced between the two non-probe vertices, called an *enhanced edge,* for the C_4 to be completed into chordal graph.



- Let G be a graph, the *enhanced graph* G* is the graph G together with all the enhanced edges.

Chordal probe - C_4 enhancing

- Let G be a graph containing no induced chordless cycles C_k, for $k>4$:

  Lemma 2[GL04][Z94] – If G has a probe/non-probe partition in which probes and non-probes alternate on every chordless 4-cycle, then the enhanced graph G* is a chordal completion of G.

- *Theorem 3* – If G is a C_k-free graph for $k>4$, then G is chordal probe $\Leftrightarrow$ G* is chordal.

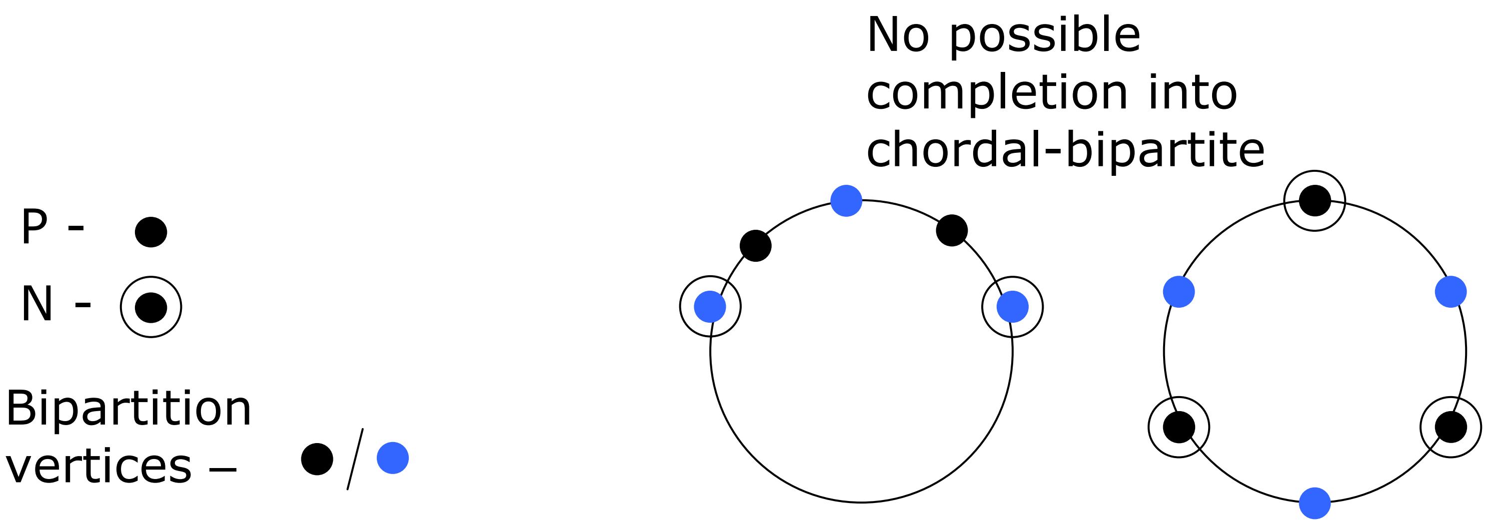
Chordal-bipartite probe graphs

- A graph $G=(V,E)$ is called *chordal bipartite probe* if
  1. $\exists$ partition $V=P(\text{probes}) \cup N(\text{non-probes})$, N is a stable set
  2. $\exists$ completion $E' \subseteq \{(u,v) \mid u,v \in N$, u and v are in different biparts$\}$

  such that $G'=(V,E \cup E')$ is bipartite.
- Since G_p is a chordal bipartite graph and N is a stable set, clearly G is a bipartite graph. Since there is only one bipartition, we keep the same bipartition by adding edge only between vertices from different biparts.

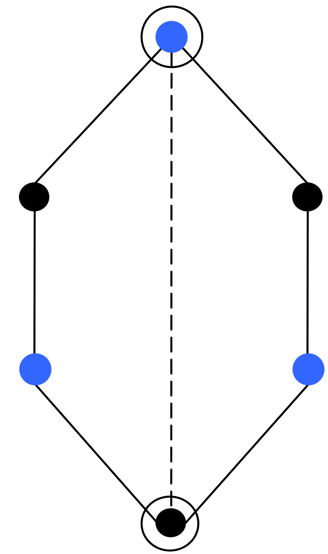
Chordal-bipartite probe – a necessary condition

- *Lemma* 4 [BCGLPSS07] – In every induced C_k, for $k \geq 6$, of a chordal-bipartite probe graph the following properties must hold:
  1. Every probe sees at most one other probe.
  2. There is at least one edge of the cycle whose endpoints are probes.



Chordal-bipartite probe - C_6 case

- C_6 is a special case of chordal-bipartite probe graphs where:
  1. There are exactly two non-probes, one black and the other blue, opposite each other.
  2. An edge is <u>forced</u> between the two non-probe vertices, called a *bi-enhanced edge,* for C_6 to be completed into a chordal-bipartite graph.



P -

N -

Bi-enhanced edge - -----

Bipartition vertices -

- Let B be a bipartite graph, the *bi-enhanced graph* B* is the graph B together with all the bi-enhanced edges.

Chordal-bipartite probe - C_6 enhancing

- Let B be a bipartite graph with no induced chordless cycles C_k, for $k>6$:

  ***Lemma* 5** – If B has a probe/non-probe partition in which there are exactly two non-probes opposite each other on every chordless 6-cycle, then the bi-enhanced graph B^* is a chordal-bipartite completion of B.

  Proof:

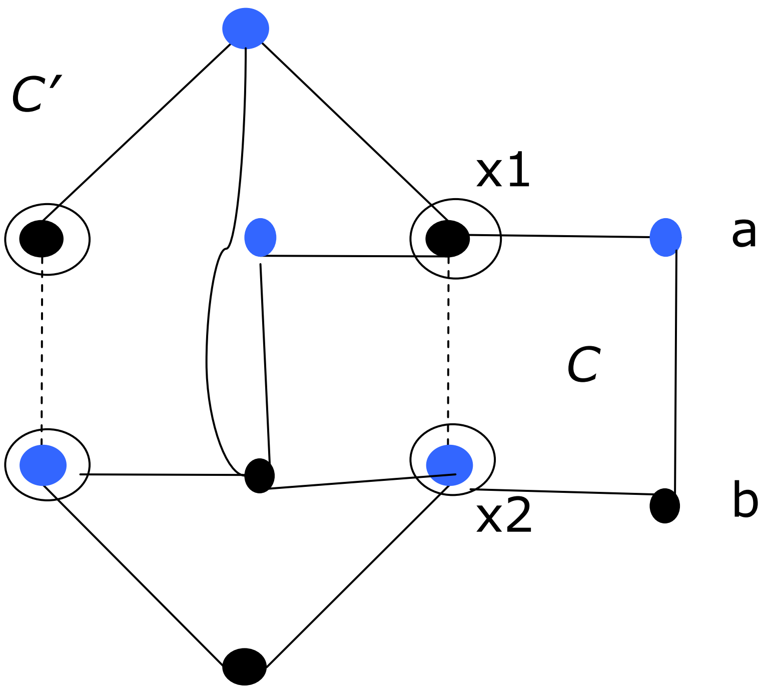
Suppose B^* is not a chordal-bipartite graph.

Then B^* is bipartite graph that has a cycle of size > 4.

Choose C' to be such a cycle in with minimal number h of bi-enhanced edges.

Proof of Lemma 5 –cont.

- *h*=0: *C′* is a cycle of size >4 in *G*, a contradiction.
- *h>0:*



Let [x1,a,b,x2] be a chordless path in B*, where a,b are probes

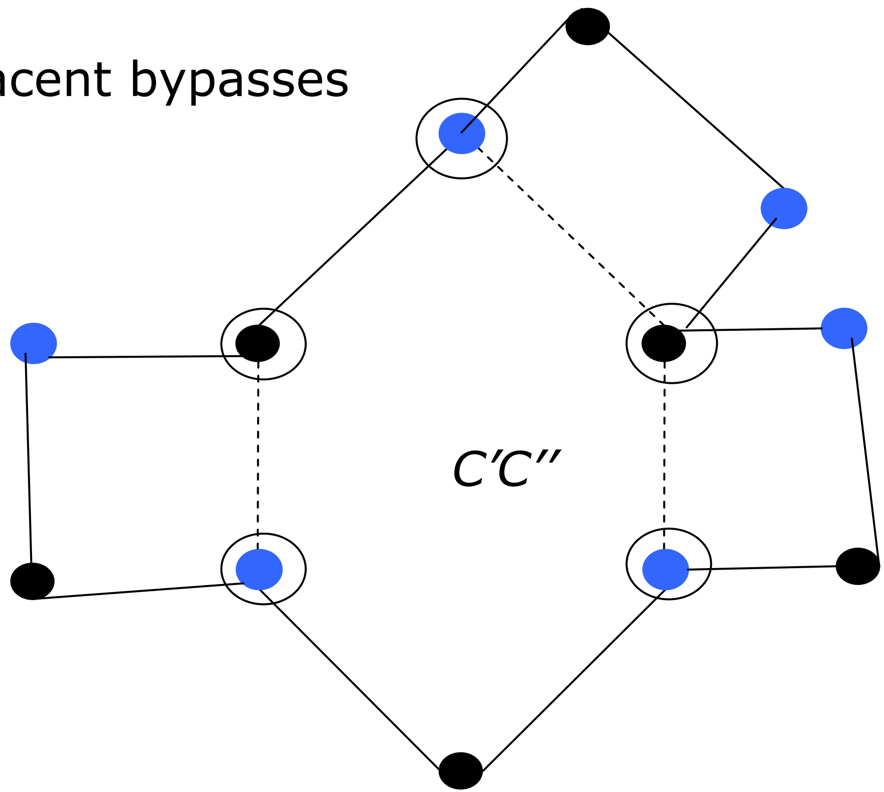
The pair {a,b} is a **bypass** of the edge (x1,x2) if and only if a,b not on C′ and have no other neighbors on C′.

Claim 1: Every bi-enhanced edge on C′ has a bypass.

Claim 2: a vertex can be in at most one bypass on C′.

Proof of Lemma 5 –cont.

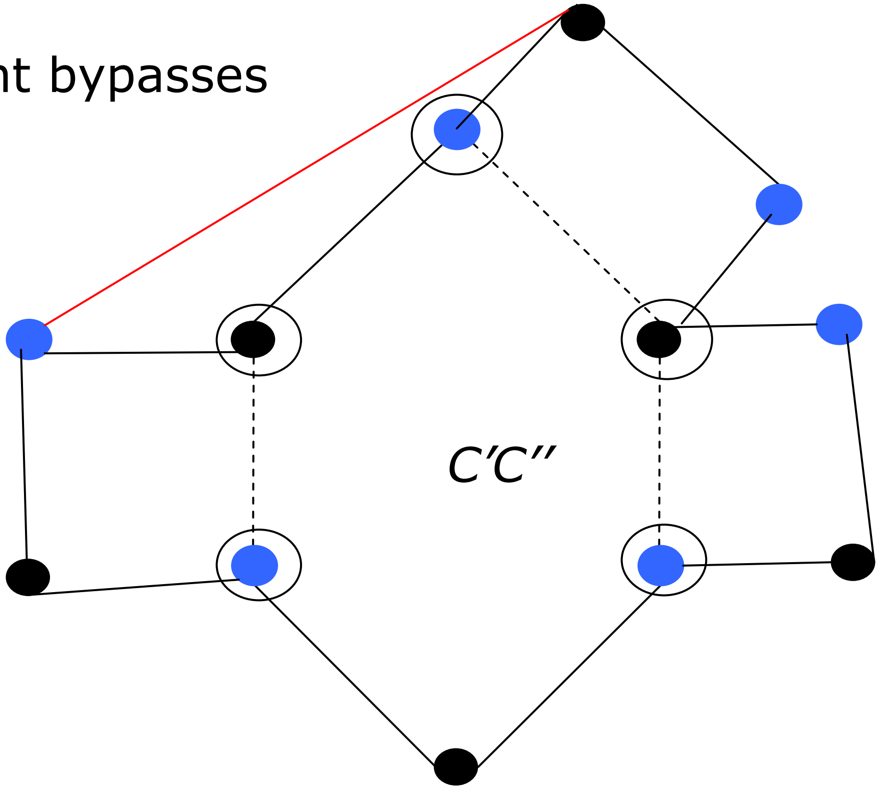
Case 1: no adjacent bypasses



C′′ is a chordless cycle of size >4 in G

Proof of Lemma 5 –cont.

Case 2: adjacent bypasses

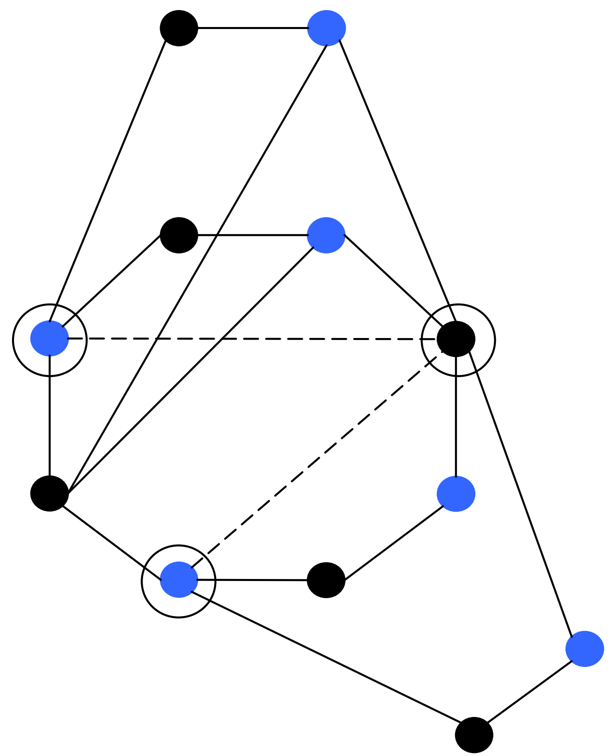


C′′ is a chordless cycle of size >4 in G

Chordal-bipartite probe - C_6 enhancing

- *Theorem* 6 – If B is a bipartite C_k-free graph for $k>6$, then B is chordal-bipartite probe $\Leftrightarrow$ B* is chordal-bipartite.

Chordal-bipartite probe - example



P -

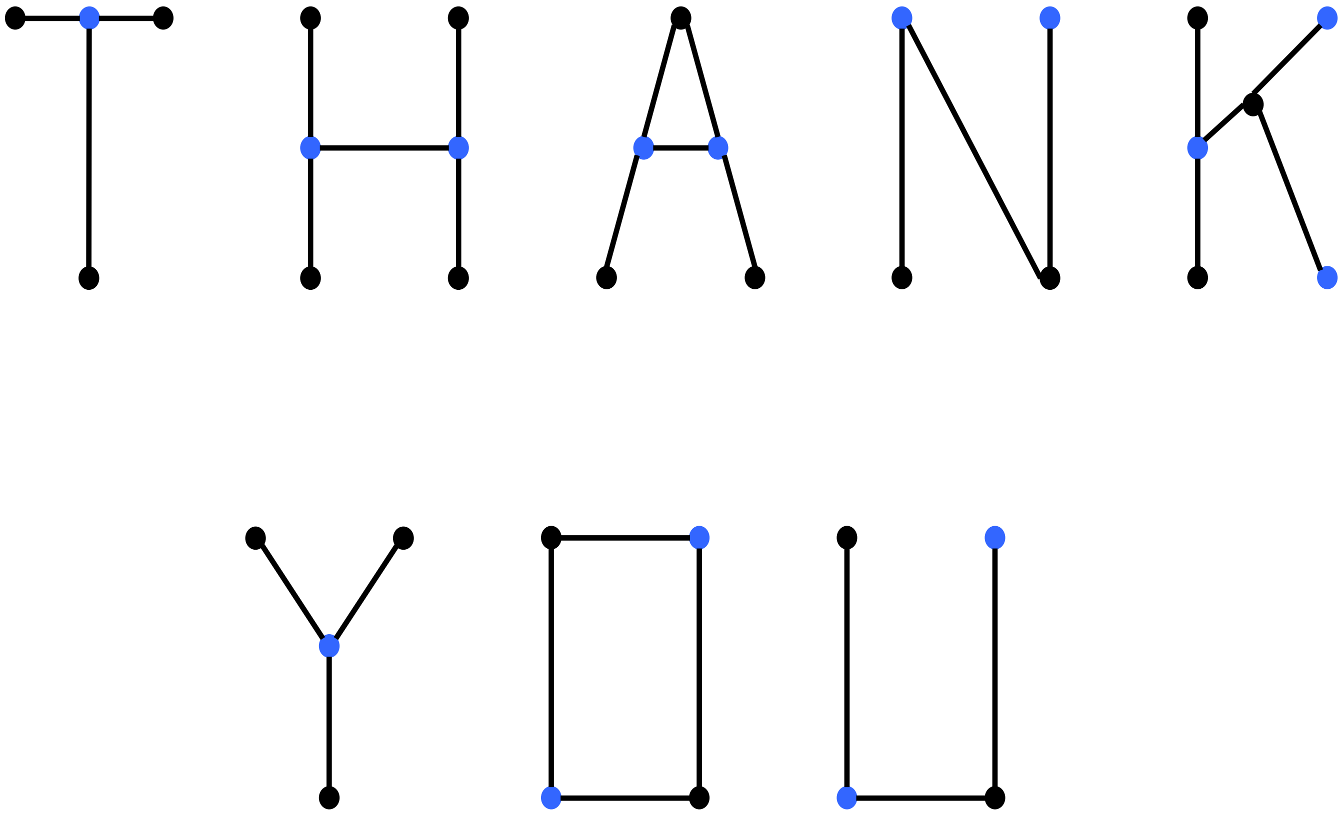
N -

Bipartition –
black/red vertices

Bi-enhanced
edge -

Open questions

- Find necessary and sufficient conditions for completing C_k into a chordal-bipartite graph, for $k>6$.
- Does a chordal-bipartite probe graph have a vertex or edge elimination algorithm ?
- Research the probe problem on other graph classes.


THANK
YOU