Edge fault-diameter of Cartesian graph bundles

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Introduction

Design of large interconnection networks

Usual constraints:

- each processor can be connected to a limited number of other processors
- the delays in communication must not be too long

Extensively studied network topologies in this context include graph products and bundles.

- an interconnection network should be fault-tolerant (some nodes or links are faulty)

The (edge) fault-diameter has been determined for many important networks.

Previous work on (edge) fault-diameters

- M. Krishnamoortthy, B. Krishnamurty: Fault diameter of interconnection networks (1987)
- K. Day, A. Al-Ayyoub: Minimal fault diameter for highly resilient product networks (2000)
- M. Xu, J.-M. Xu, X.-M. Hou: Fault diameter of Cartesian product graphs (2005)
- Banič, Žerovnik: Fault-diameter of Cartesian graph bundles (2006), Edge fault-diameter of Cartesian product of graphs (2007), Fault-diameter of Cartesian product of graphs (2008)

Cartesian product of graphs

Definition 2. Let G_1 and G_2 be graphs. The *Cartesian product* of graphs G_1 and G_2, $G = G_1 \Box G_2$, is defined on the vertex set $V(G_1) \times V(G_2)$. Vertices (u_1, v_1) and (u_2, v_2) are adjacent if either $u_1u_2 \in E(G_1)$ and $v_1 = v_2$ or $v_1v_2 \in E(G_2)$ and $u_1 = u_2$.

Cartesian graph bundle

Definition 3. Let B and F be graphs. A graph G is a *Cartesian graph bundle with fibre* F *over the base graph* B if there is a *graph map* $p : G \to B$ such that for each vertex $v \in V(B)$, $p^{-1}(\{v\})$ is isomorphic to F, and for each edge $e = uv \in E(B)$, $p^{-1}(\{e\})$ is isomorphic to $F \Box K_2$.

- The mapping p is also called the *projection* (of the bundle G to its base B).

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- The mapping p is also called the *projection* (of the bundle G to its base B).
- We say an edge $e \in E(G)$ is *degenerate* if $p(e)$ is a vertex. Otherwise we call it *nondegenerate*.
- Note that each edge $e = uv \in E(B)$ naturally induces an isomorphism $\varphi_e : p^{-1}(\{u\}) \to p^{-1}(\{v\})$ between two fibres.

Example 1

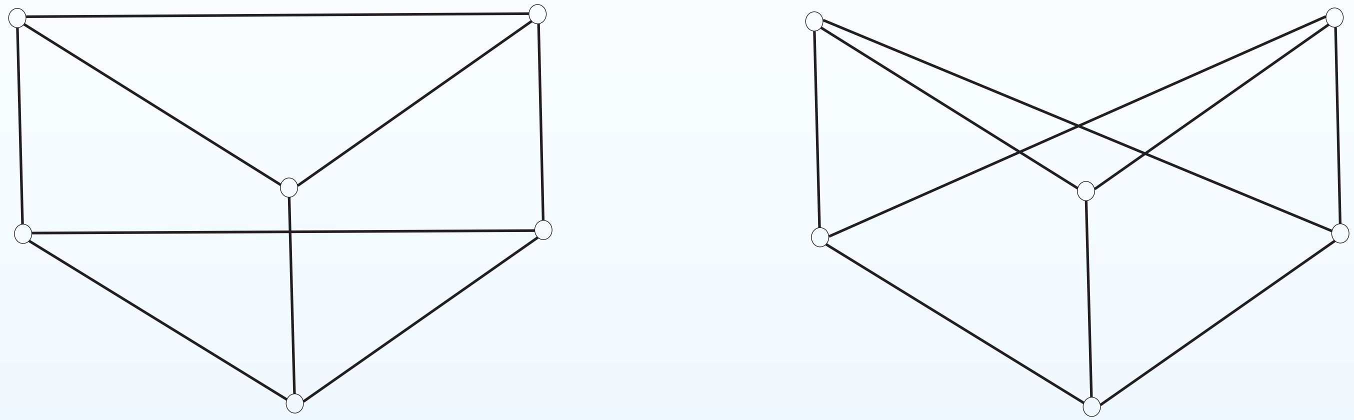

Figure 1: Nonisomorphic bundles

Let $F = K_2$ and $B = C_3$. On Figure 1 we see two nonisomorphic bundles with fibre F over the base graph B.
Informally, one can say that bundles are "twisted products".

Example 2

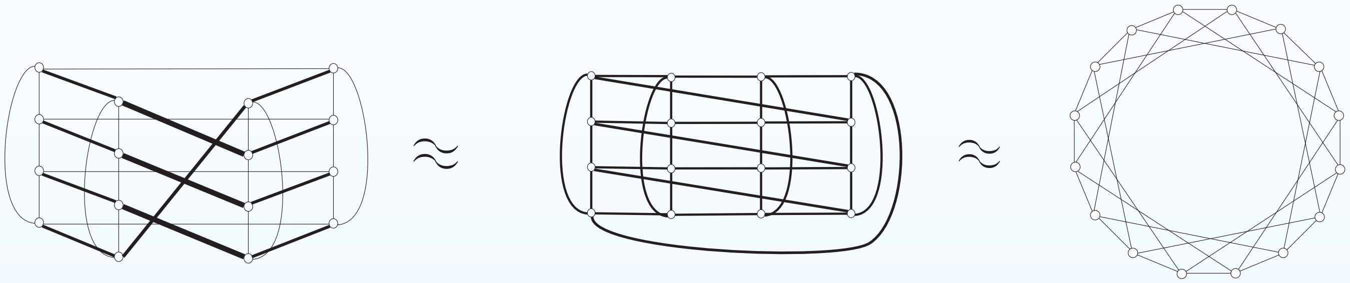

Figure 2: Twisted torus: Cartesian graph bundle with fibre C_4 over base C_4.

It is less known that graph bundles also appear as computer topologies. A well known example is the twisted torus on Figure 2. Cartesian graph bundle with fibre C_4 over base C_4 is the ILIAC IV architecture.

Edge fault-diameter

Definition 4. The *edge-connectivity* of a graph G, $\lambda(G)$, is the minimum cardinality over all edge-separating sets in G. A graph G is said to be k-*edge connected*, if $\lambda(G) \geq k$.

Edge fault-diameter

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Definition 5. Let G be a k-edge connected graph and $0 \leq a < k$. Then we define the a-*edge fault-diameter* of G as

$$\bar{\mathcal{D}}_a(G) = \max\{\mathsf{d}(G \setminus X) \mid X \subseteq E(G), |X| = a\}.$$

- Note that $\bar{\mathcal{D}}_a(G)$ is the largest diameter among subgraphs of G with a edges deleted, hence $\bar{\mathcal{D}}_0(G)$ is just the diameter of G, $\mathsf{d}(G)$.

Results 1

Theorem 1. Let F and B be k_F-edge connected and k_B-edge connected graphs respectively, and G a Cartesian graph bundle with fibre F over the base graph B. Let $\lambda(G)$ be the edge-connectivity of G. Then $\lambda(G) \geq k_F + k_B$.

Results 1

Theorem 1. Let F and B be k_F-edge connected and k_B-edge connected graphs respectively, and G a Cartesian graph bundle with fibre F over the base graph B. Let $\lambda(G)$ be the edge-connectivity of G. Then $\lambda(G) \geq k_F + k_B$.

Corollary 2. Let G_1 and G_2 be k_1 and k_2-edge connected graphs, respectively. Then the Cartesian product $G_1 \Box G_2$ is at least $(k_1 + k_2)$-edge connected.

Results 2

Theorem 3. Let F and B be k_F-edge connected and k_B-edge connected graphs respectively, $0 \leq a < k_F$, $0 \leq b < k_B$, and G a Cartesian bundle with fibre F over the base graph B. Then

$$\bar{\mathcal{D}}_{a+b+1}(G) \leq \bar{\mathcal{D}}_a(F) + \bar{\mathcal{D}}_b(B) + 1.$$

Example 3

Let $G = K_2 \Box K_2$, see Figure 3. G is a graph bundle with fiber $F = K_2$ over the base graph $B = K_2$. Then for $a = b = 0$ we have

$$\bar{\mathcal{D}}_{a+b+1}(G) = 3,$$

$$\bar{\mathcal{D}}_b(B) + \bar{\mathcal{D}}_a(F) + 1 = 1 + 1 + 1 = 3.$$

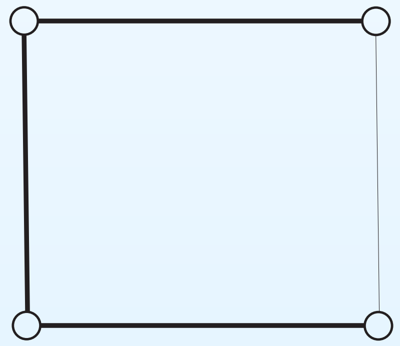

Figure 3: $G = K_2 \Box K_2$ with one faulty link.

Future Work

- Let G be a k connected graph and $0 < a < k$. Then

$$\bar{\mathcal{D}}_a(G) \leq D_a(G) + 1$$

- *Mixed fault-diameter* of G, $D_{(m,n)}(G)$.

  Let G be a k connected graph and $0 < a < k$, $m + n = a$. Then

$$D_{(m,n)}(G) \leq D_{(m-l,n+l)}(G), \quad l < m$$

  and

$$\bar{\mathcal{D}}_a(G) \leq D_{(m,n)}(G) \leq D_a(G) + 1, \quad m \neq 0$$