ON TOTAL CHROMATIC NUMBER OF DIRECT PRODUCT GRAPHS

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Cartesian product

Definition

Cartesian product of graphs G and H is graph $G \square H$ defined on

$$V(G \square H) = V(G) \times V(H)$$

$$E(G \square H) = \{(u,v)(x,y) | u = x, vy \in E(H), \text{ or }, v = y, ux \in E(G)\}$$

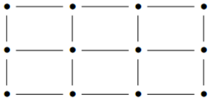


Figure: Cartesian product $P_4 \square P_3$

Direct product

Definition

Direct product of graphs G and H is graph $G \times H$ defined on

$$V(G \times H) = V(G) \times V(H)$$

$$E(G \times H) = \{(u, v)(x, y) | ux \in E(G) \text{ and } vy \in E(H)\}$$

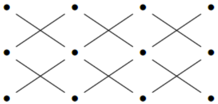


Figure: Direct (tensor) product $P_4 \times P_3$

Strong product

Definition

Strong product of graphs G and H is graph $G \boxtimes H$ defined on

$$V(G \boxtimes H) = V(G) \times V(H)$$

$$E(G \boxtimes H) = \{(u, v)(x, y) | ux \in E(G), \text{ and,}$$

$$vy \in E(H), \text{ or, } u = x, vy \in E(H), \text{ or,} v = y, ux \in E(G)\}$$

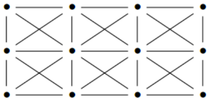


Figure: Strong product $P_4 \boxtimes P_3$

Lexicographic product

Definition

Lexicographic product of graphs G and H is graph $G \bullet H$ defined on

$$V(G \bullet H) = V(G) \times V(H)$$

$$E(G \bullet H) = \{(u,v)(x,y) | ux \in E(G), \text{ or, } u = x, vy \in E(H)\}$$

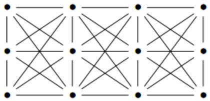
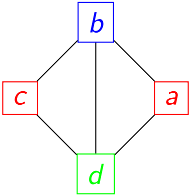


Figure: Lexicographic product $P_4 \bullet P_3$

Graph coloring

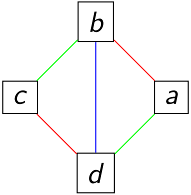
Vertex coloring is a mapping $f : V(G) \rightarrow C = \{1, 2, ..., n\}$ such that $uv \in E(G)$ implies $f(u) \neq f(v)$.



Smallest n for which such coloring exists is called **chromatic number**, $\chi(G)$.

Graph coloring

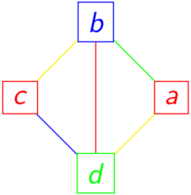
Edge coloring is a mapping $f : E(G) \rightarrow C' = \{1, 2, ..., n\}$ such that incident edges do not share same color.



Smallest n for which such coloring exists is called **edge chromatic number**, $\chi'(G)$.

Graph coloring

Total coloring is a mapping $f : V(G) \cup E(G) \rightarrow C'' = \{1, 2, ..., n\}$ such that any two elements that are either adjacent or incident are assigned different colors.



The minimum number of colors needed for a proper total coloring is the **total chromatic number** of G, denoted by $\chi''(G)$ or $\chi_T(G)$.

Some results on coloring of direct products

- *Hedetniemi's conjecture(1966)*:
  $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$.
- *Greenwell,Lovasz(1974)*:
  G connected graph with $\chi(G) > n$: $\chi(G \times K_n)$ is uniquely n-colorable.
- *Welzl(1984); Duffus,Sands,Woodrow(1985)*:
  G,H connected, $(n+1)$-chromatic graphs containing a complete subgraph: $\chi(G \times H) = n + 1$.

Some results on total coloring

Conjecture

Total coloring conjecture (Behzad, Vizing)
For every graph G,

$$\chi''(G) \leq \Delta(G) + 2.$$

- (Vijayatidya)
  G graph with maximum degree 3: $\chi''(G) \leq 5$.
- (Zmazek, Žerovnik(2004))
  G,H arbitrary graphs, $\Delta(G) \leq \Delta(H) : \chi''(G \Box H) \leq \Delta(G) + \chi''(H)$.
- (Campos, Mello (2007)
  C_k^n, $n \equiv r(\mod k+1)$, n even and $r \neq 0 : \chi''(C_k^n) \leq \Delta(C_k^n) + 2$

Total chromatic number of $G \times P_n$)

Theorem

$\chi''(G \times P_n) = \Delta(G \times P_n) + 1,\ \textit{if}\ \chi'(G) = \Delta(G)$

Proof.

$$\varphi : S \rightarrow C''$$

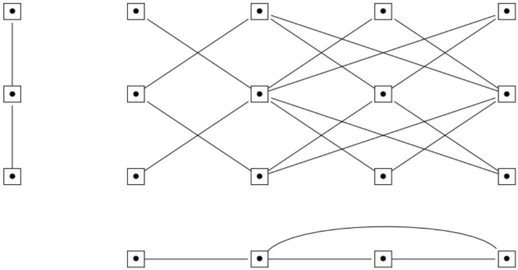
$$\varphi((g,h)) = \chi'(G) \cdot (C(h)+1)(\text{mod}\Theta)$$

$$\varphi((g,h),(g',h')) = C'(g,g') + \chi'(G) \cdot C(h)(\text{mod}\Theta);\ h < h'$$

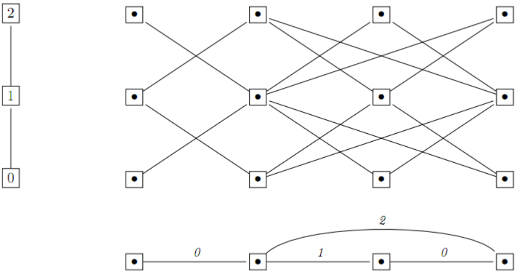
where $\Theta = (\Delta(G) \cdot 2 + 1)$ and $C'' = \{0, 1, 2, ..., \Theta - 1\}$.

□

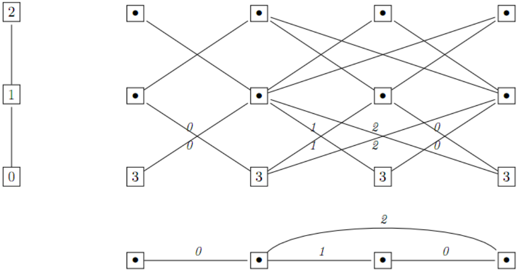
Sketch of coloring



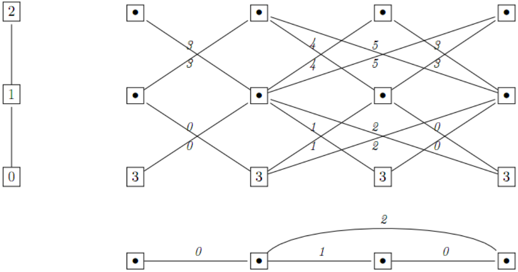
Sketch of coloring



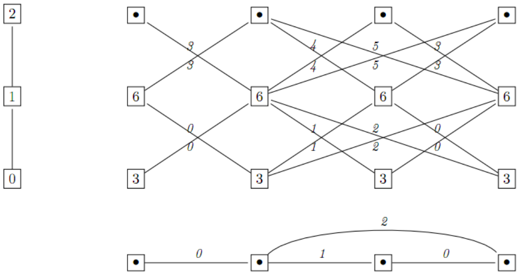
Sketch of coloring



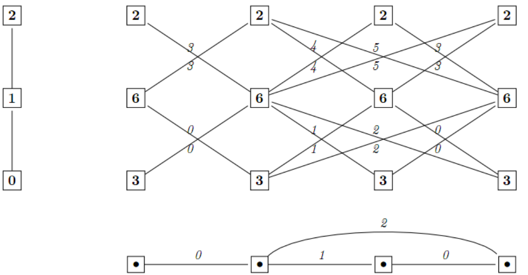
Sketch of coloring



Sketch of coloring



Sketch of coloring



Colloraries

Remark

The function used in the proof will also produce total coloring of an arbitrary graph $G \times P_n$, if the division is done by $\Theta = \Delta(G) \cdot 2 + 3$ and $\chi'(G) = \Delta(G) + 1$. However it will use $\Delta(G) \cdot 2 + 3$ colors which does not match the conjecture and is not the proper coloring.Better colorings exist in this case.

Collorary

$$\begin{aligned} \chi''(P_n \times P_m) &= 5 \\ &= \Delta(P_n \times P_m) + 1 \end{aligned}$$

Total chromatic number of $C_m \times P_n$

Lemma

For even cycle C_{2k}, there exists total coloring with most 4 colors.

Theorem

$\chi''(C_m \times P_n) = 5$

Sketch of coloring

Future work

- $\chi''(C_m \times C_n) = ?(5)$
- $\chi''(G \times C_n) = ?$
- $\chi''(G \times K_n) = ?$
- $\chi''(G \times H) = ?$