Using latin squares to color split graphs

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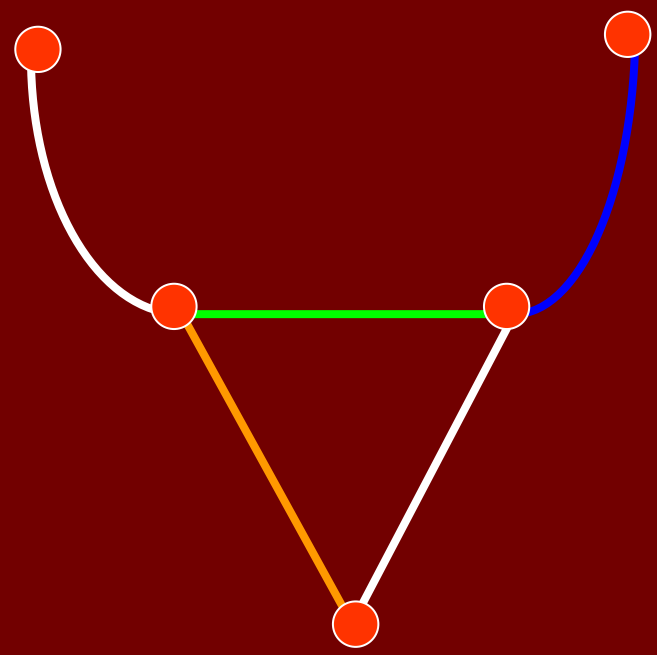
Outline

- Edge-Coloring
- The Classification Problem
- Split Graphs
- Overfull Graphs
- The Edge-Coloring Conjecture for Split Graphs
- Latin Squares
- The Edge-Coloring of Split Graphs Using Latin Squares

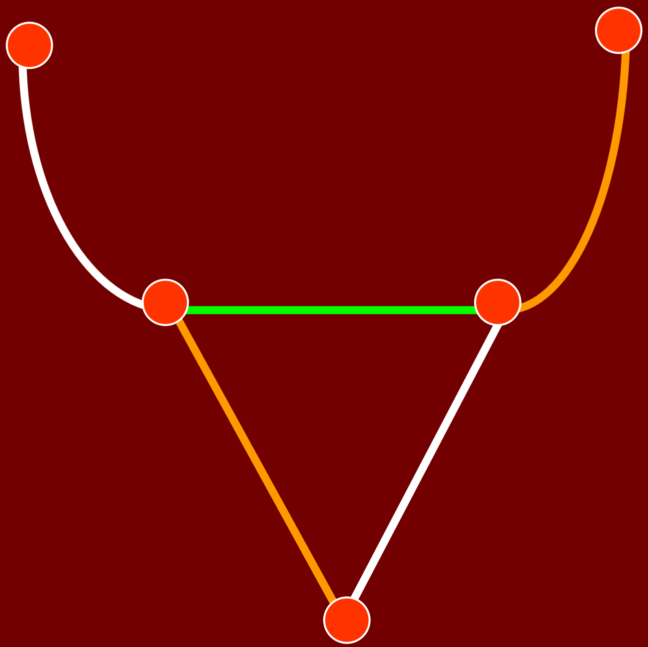
Edge-Coloring

A ***k-edge-coloring*** of a graph G is an assignment of k colors to the edges of G such that any two edges incident in a common vertex have distinct colors.



The Edge-Coloring Problem

The minimum k required to perform a k-edge-coloring of a simple graph G is called ***chromatic index*** of G and is denoted by $\chi'(G)$.

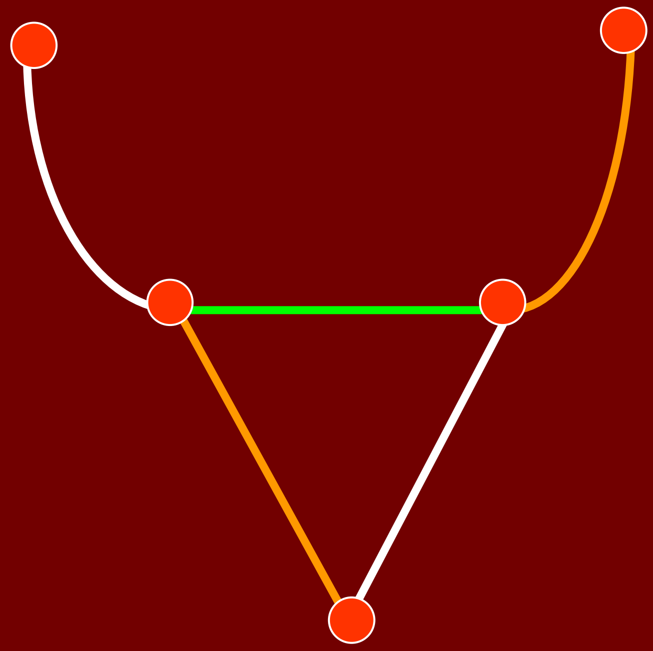


$\chi'(\text{Bull})=3$

The Edge-Coloring Problem

$\chi'(G) \geq \Delta(G)$,

where $\Delta(G)$ is the maximum degree of a graph G.



$\chi'(\text{Bull})=3$

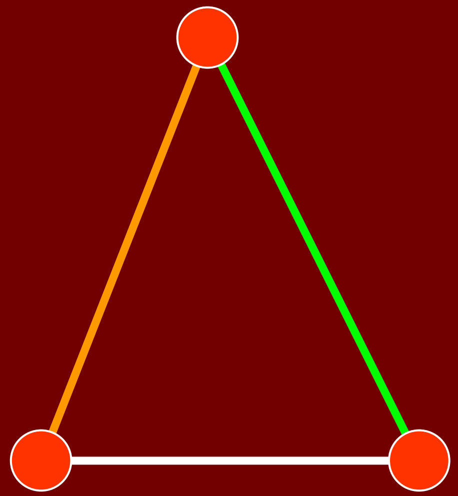
The Edge-Coloring Problem

There are graphs that have

$\chi'(G) > \Delta(G)$.

In 1964, Vizing showed that every simple graph G has

$\chi'(G) \leq \Delta(G)+1$.



$\Delta(C_3)=2$ $\chi'(C_3) = 3$

The Edge-Coloring Problem

A direct result of the Vizing's Theorem is that any simple graph G has

$$\Delta(G) \leq \chi'(G) \leq \Delta(G)+1.$$

Vizing restricted the Edge-Coloring Problem to the following problem:

Given a simple graph G, is the chromatic index of G equal to $\Delta(G)$ or $\Delta(G)+1$?

The Classification Problem

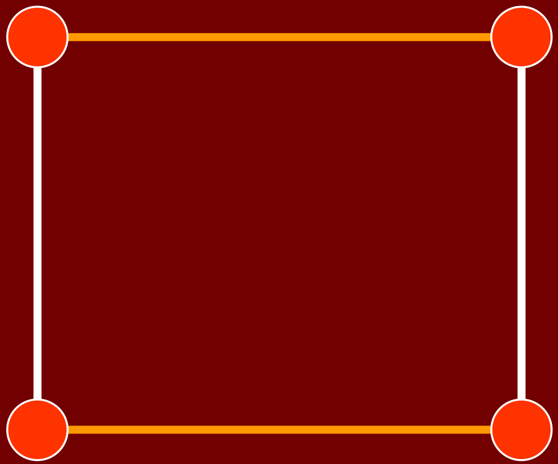
Given a simple graph G, is the chromatic index of G equal to $\Delta(G)$ or $\Delta(G)+1$?

This problem is known as the ***Classification Problem***.

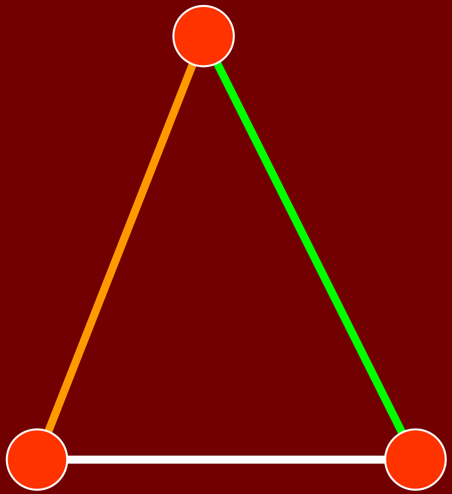
The Classification Problem

If $\chi'(G) = \Delta(G)$, then G is Class 1.

Otherwise, $\chi'(G) = \Delta(G) + 1$ and G is Class 2.



Classe 1



Classe 2

The Classification Problem

In 1981, Holyer showed that the problem of deciding if a simple graph G is Class 1 is NP-Complete.

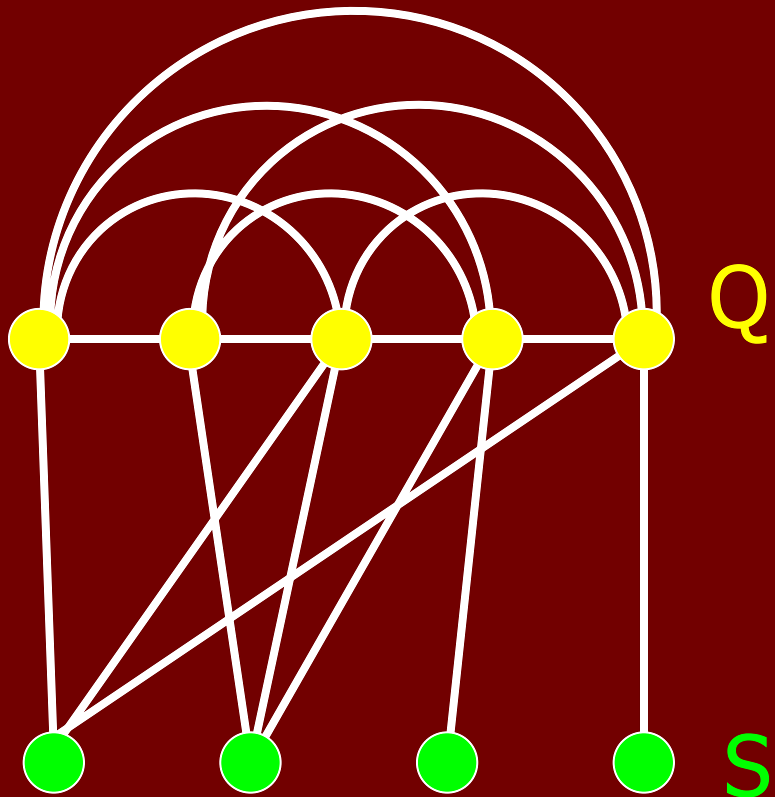
The Classification Problem

Even so, there are efficient algorithms to solve this problem when we are restricted to some classes of graphs.

- bipartite graphs are Class 1.
- a complete graph is Class 1 iff it has an even number of vertices.
- a cycle (without chords) is Class 1 iff it has an even number of vertices.
- split graphs with odd maximum degree are Class 1.

Split Graphs

A graph G is a ***split graph*** if the set of vertices of G admits a partition [Q, S], where Q is a clique and S is a stable set.

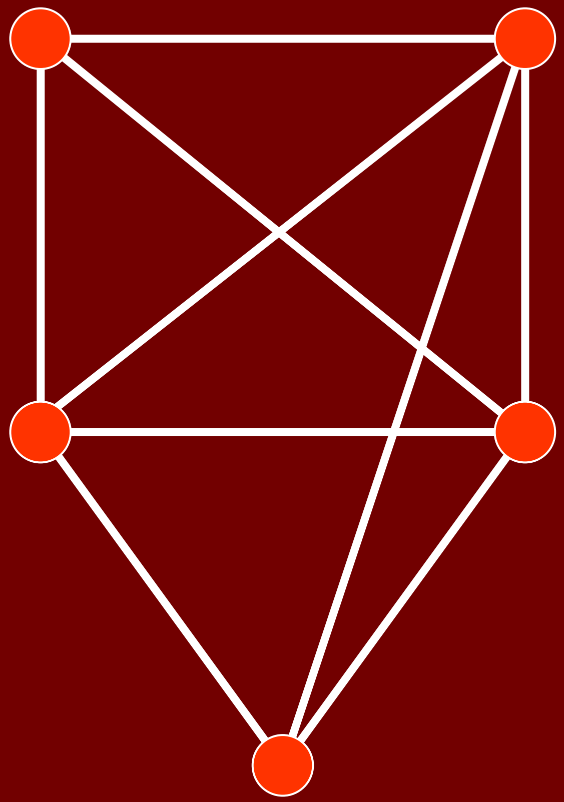


Overfull Graphs

Consider a graph G, with n vertices and m edges.

The graph G is ***overfull*** if n is odd and

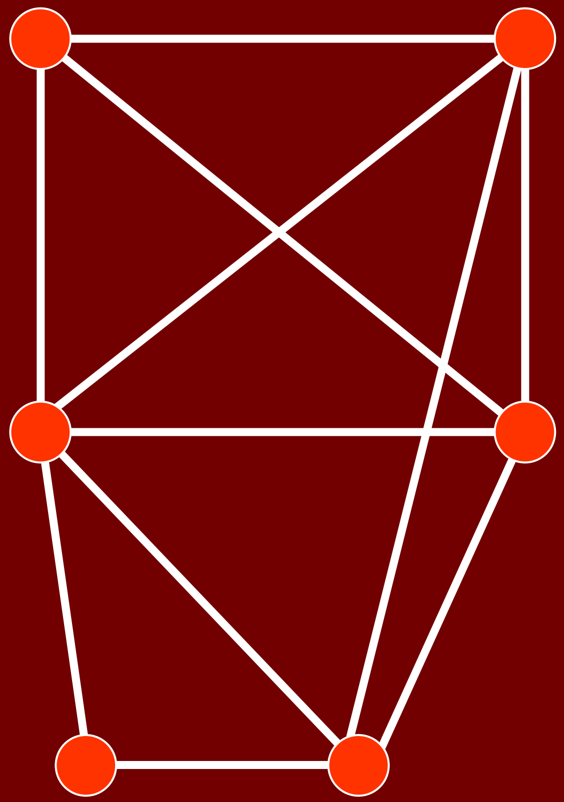
$m > \Delta(G) \lfloor n/2 \rfloor$.



$m = 9 > \Delta(G) \lfloor n/2 \rfloor =$
$= 4 \lfloor 5/2 \rfloor = 8$

Subgraph-Overfull Graphs

If G has a subgraph which is overfull and has maximum degree equal to $\Delta(G)$, then G is ***subgraph-overfull***.



Subgraph-Overfull Graphs

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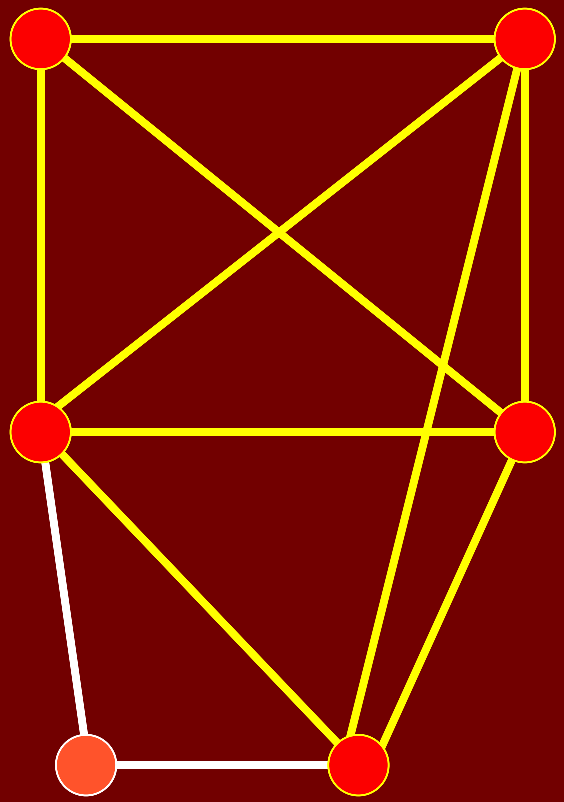
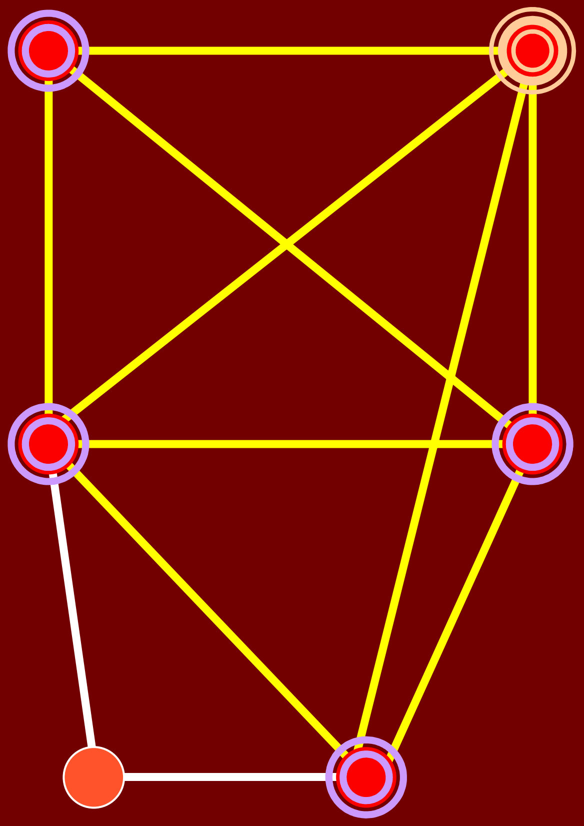


$$m = 9 > \Delta(G)\lfloor n/2 \rfloor =$$
$$= 4\lfloor 5/2 \rfloor = 8$$

Neighborhood-Overfull Graphs

If G has an overfull subgraph induced by a vertex with degree $\Delta(G)$ and its neighbors, then G is ***neighborhood-overfull***.

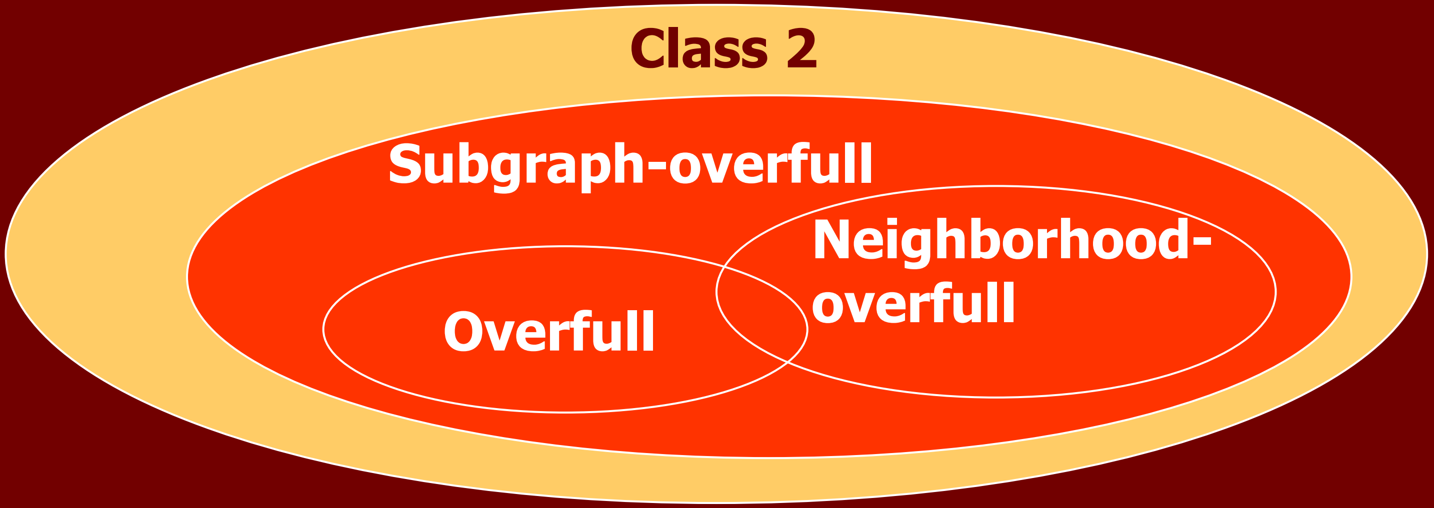


$m = 9 > \Delta(G) \lfloor n/2 \rfloor =$
$= 4 \lfloor 5/2 \rfloor = 8$

Subgraph-Overfull Graphs

Overfull graphs and neighborhood-overfull graphs are subgraph-overfull graphs.

Every subgraph-overfull graph is Class 2.



Edge-Coloring Conjecture for Split Graphs

Figueiredo, Meidanis and Mello show that every subgraph-overfull split graph is neighborhood-overfull.

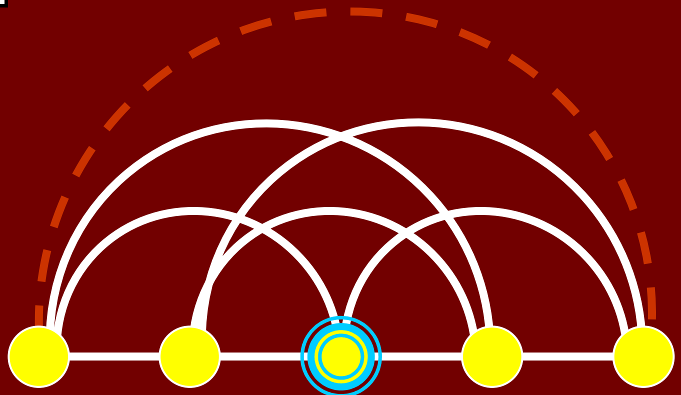
They present the following conjecture:

A split graph G is Class 2 if, and only if, G is neighborhood-overfull.

Graphs with Universal Vertices

Planthold presents the following theorem:

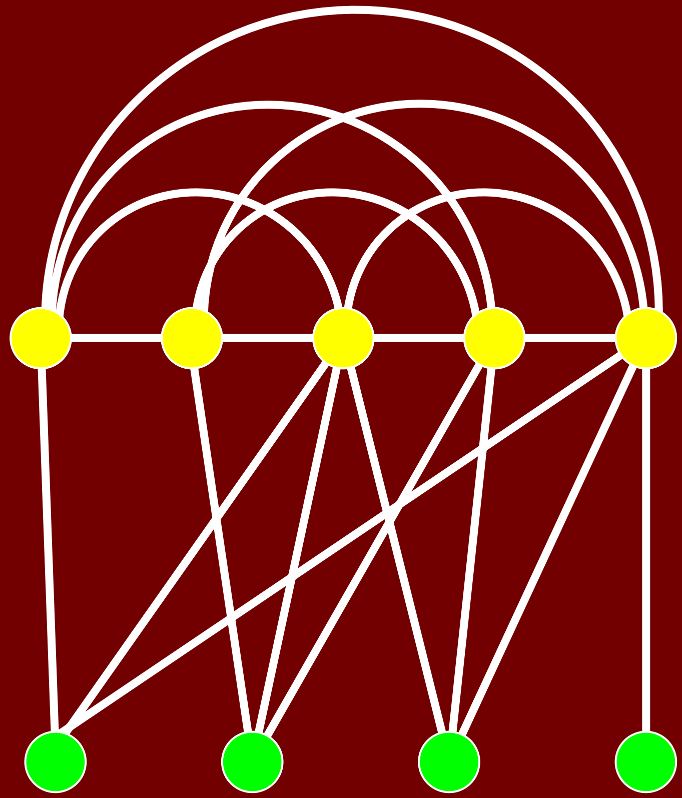
Every simple graph G containing a universal vertex is Class 2 iff G is subgraph-overfull.



K_5 minus one edge is overfull → K_5 is subgraph-overfull

Split Graphs

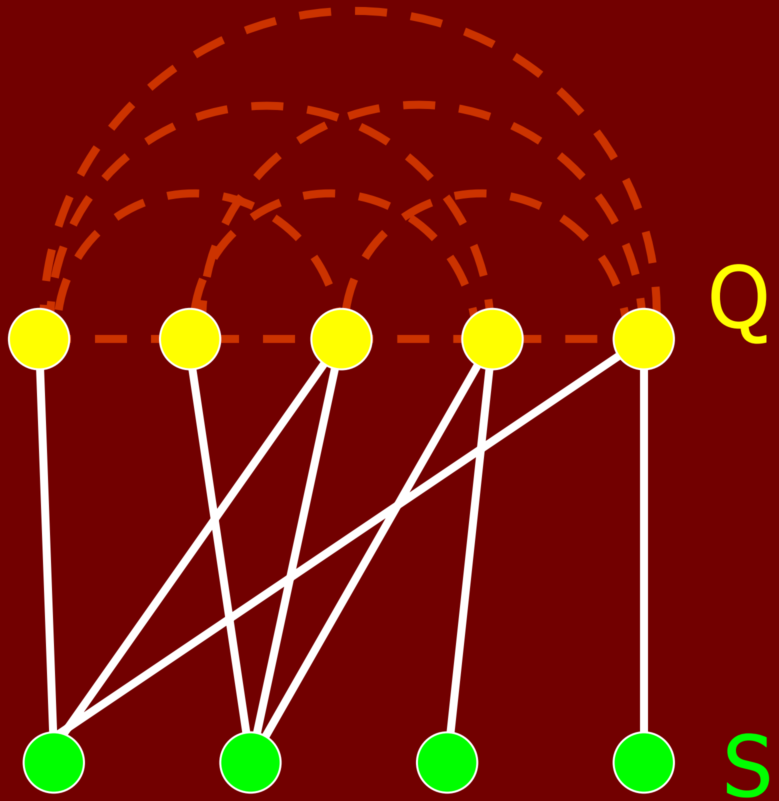
In 1995, Chen, Fu, and Ko showed that split graphs with odd $\Delta(G)$ are Class 1.



$\Delta(G)=7$

Split Graphs

Every split graph G with partition [Q, S] has a bipartite subgraph induced by the edges with a vertex in Q and another vertex in S.



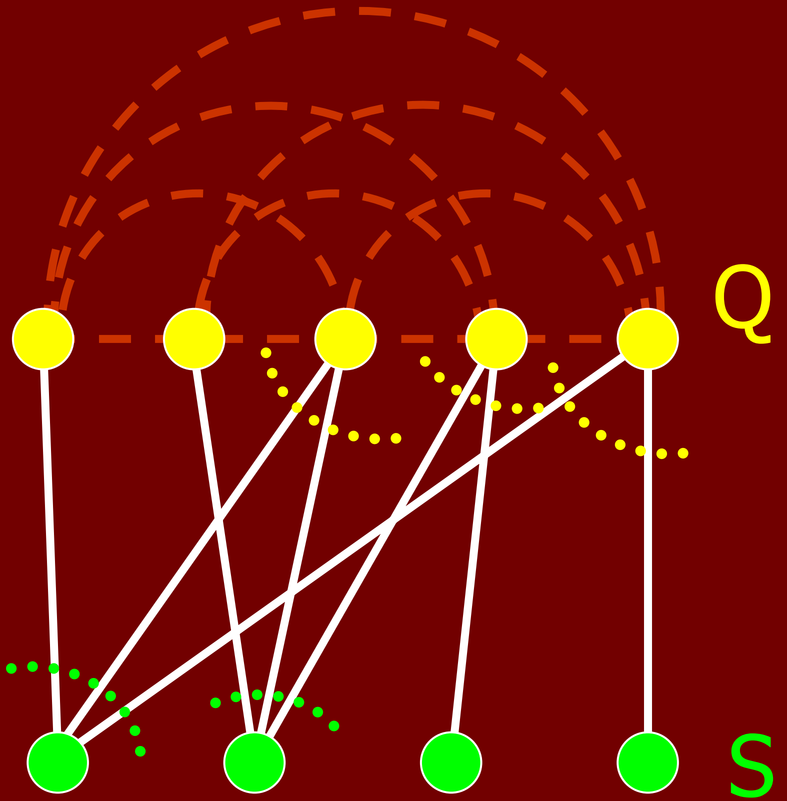
Split Graphs

Considering the bipartite
ubgraph of G, we denote:

$d(Q) = \max\{d(v), v \in Q\}$

and

$d(S) = \max\{d(v), v \in S\}$.



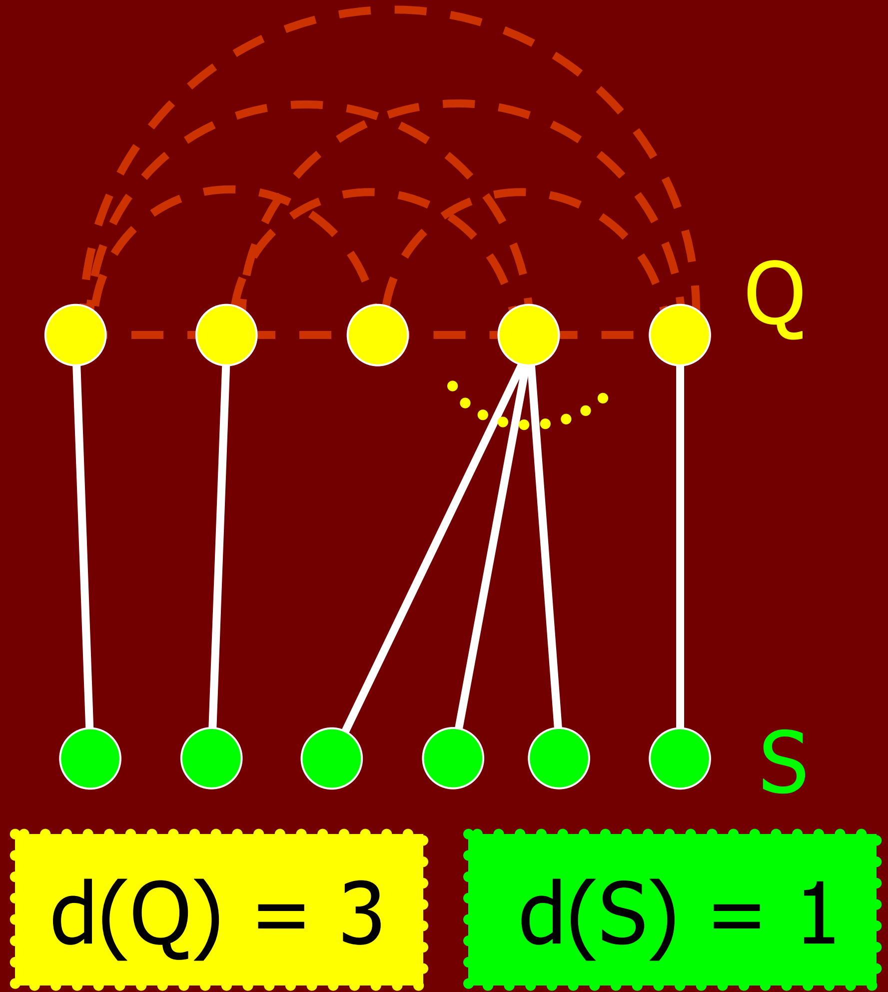
d(Q) = 2

d(S) = 3

Split Graphs

Consider the partition [Q, S] of a split graph, where Q is a maximal clique.

Chen, Fu and Ko also showed that every split graph with d(Q) $\geq$ d(S) is Class 1.



Edge-Coloring of Split Graphs

- Split Graphs that are neighborhood-overfull are Class 2.
- Split Graphs with odd maximum degree are Class 1.
- Split Graphs with even maximum degree that are not neighborhood-overfull and contain a universal vertex are Class 1.
- Split-Graphs with partition [Q, S], where Q is a maximal clique and such that $d(Q) \geq d(S)$ are Class 1.

How about split graphs with even maximum degree and $d(S) > d(Q)$, that are not neighborhood-overfull and do not contain universal vertices? Are these graphs Class 1?

Latin Square

A ***latin square of order k*** is

- a k×k-matrix
- filled with entries from {0, 1, ..., k-1}
- each element appears exactly once in each row
- each element appears exactly once in each column.

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

Commutative Latin Square

A latin square $M=[m_{i,j}]$ is ***commutative*** if

$$m_{i,j} = m_{j,i} \quad \text{for } 0 \le i,j \le k-1.$$

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

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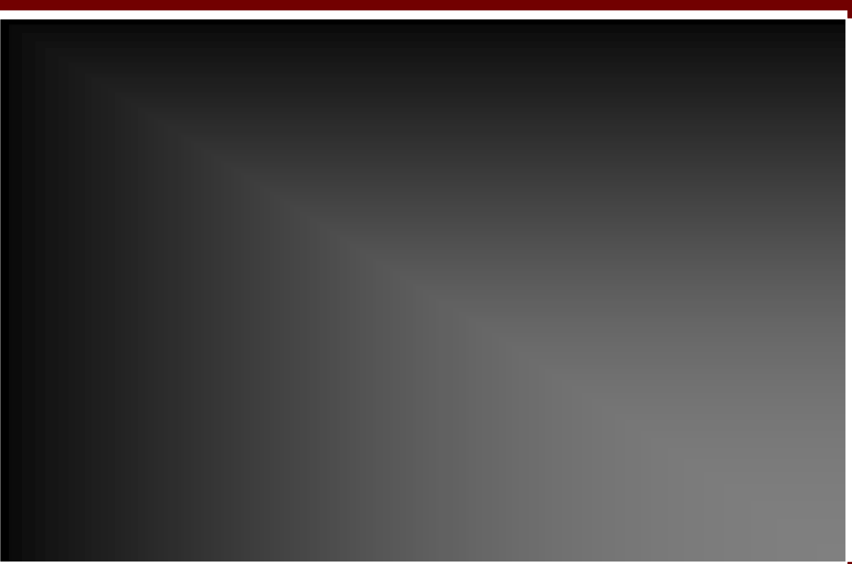

0	1	2	3	4
4	0	1	2	3
3	4	0	1	2
2	3	4	0	1
1	2	3	4	0

Commutative Latin Square

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0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

0	1	2	3	4
4	0	1	2	3
3	4	0	1	2
2	3	4	0	1
1	2	3	4	0

Idempotente Latin Square

A latin square $M=[m_{i,j}]$ is ***idempotente*** if

$$m_{i,i} = i \quad \text{for } 0 \leq i \leq k\text{-}1.$$

0	3	1	4	2
3	1	4	2	0
1	4	2	0	3
4	2	0	3	1
2	0	3	1	4

Idempotente Latin Square

A latin square $M=[m_{i,j}]$ is ***idempotente*** if

$$m_{i,i} = i \quad \text{for } 0 \leq i \leq k-1.$$

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

Idempotente Latin Square

A latin square $M=[m_{i,j}]$ is ***idempotente*** if

$$m_{i,i} = i \quad \text{for } 0 \leq i \leq k\text{-}1.$$

0	3	1	4	2
3	1	4	2	0
1	4	2	0	3
4	2	0	3	1
2	0	3	1	4

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

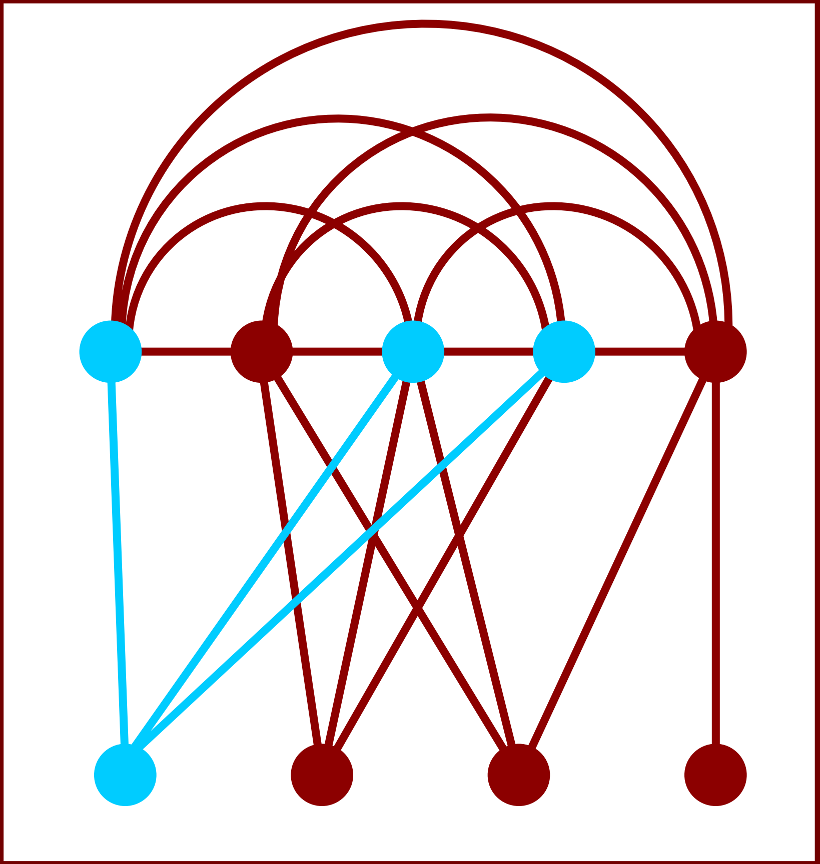
Using Latin Squares to Color Split Graphs

Chen, Fu and Ko use **idempotente commutative latin squares** to prove that split graphs with odd maximum degree are Class 1.

0	4	1	5	2	6	3
4	1	5	2	6	3	0
1	5	2	6	3	0	4
5	2	6	3	0	4	1
2	6	3	0	4	1	5
6	3	0	4	1	5	2
3	0	4	1	5	2	6

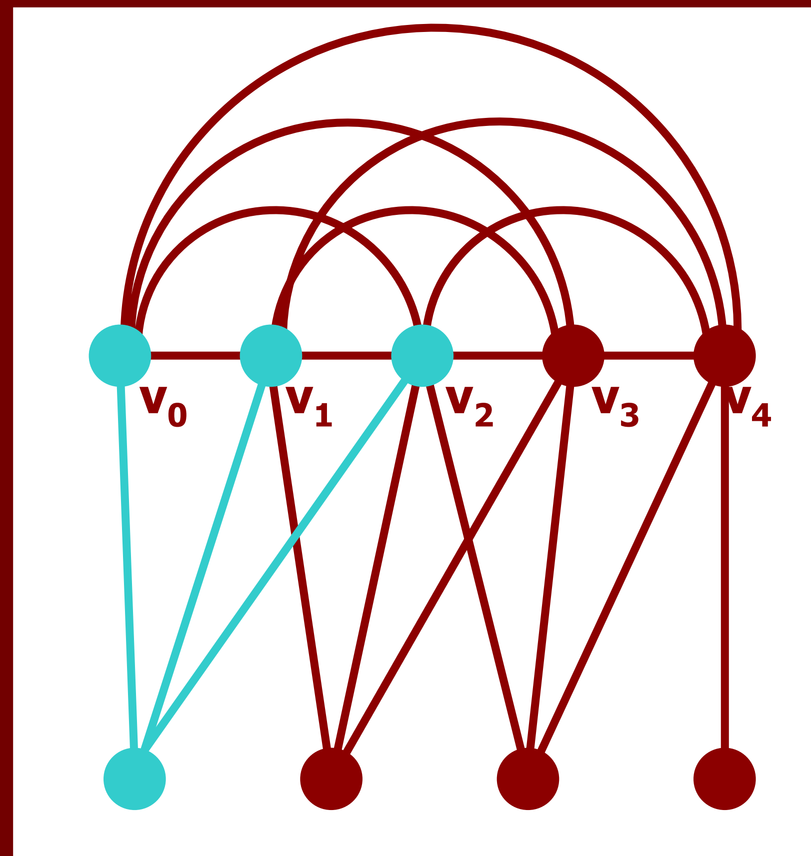
Using Latin Squares to Color Split Graphs

0	4	1	5	2	6	3
4	1	5	2	6	3	0
1	5	2	6	3	0	4
5	2	6	3	0	4	1
2	6	3	0	4	1	5
6	3	0	4	1	5	2
3	0	4	1	5	2	6



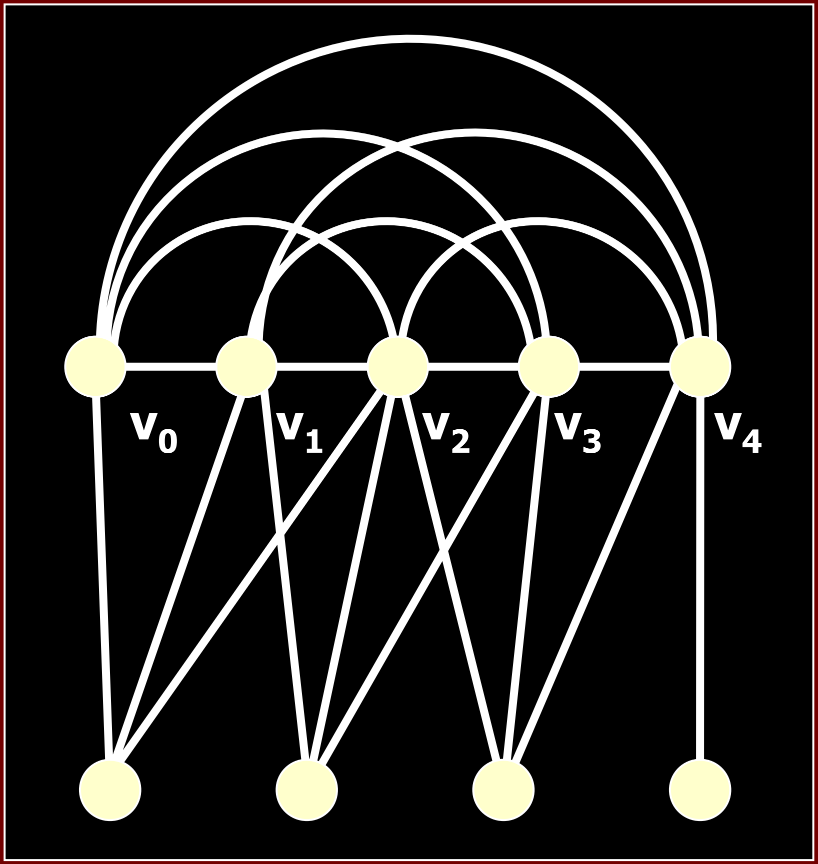
Using Latin Squares to Color Split Graphs

0	4	1	5	2	6	3
4	1	5	2	6	3	0
1	5	2	6	3	0	4
5	2	6	3	0	4	1
2	6	3	0	4	1	5
6	3	0	4	1	5	2
3	0	4	1	5	2	6



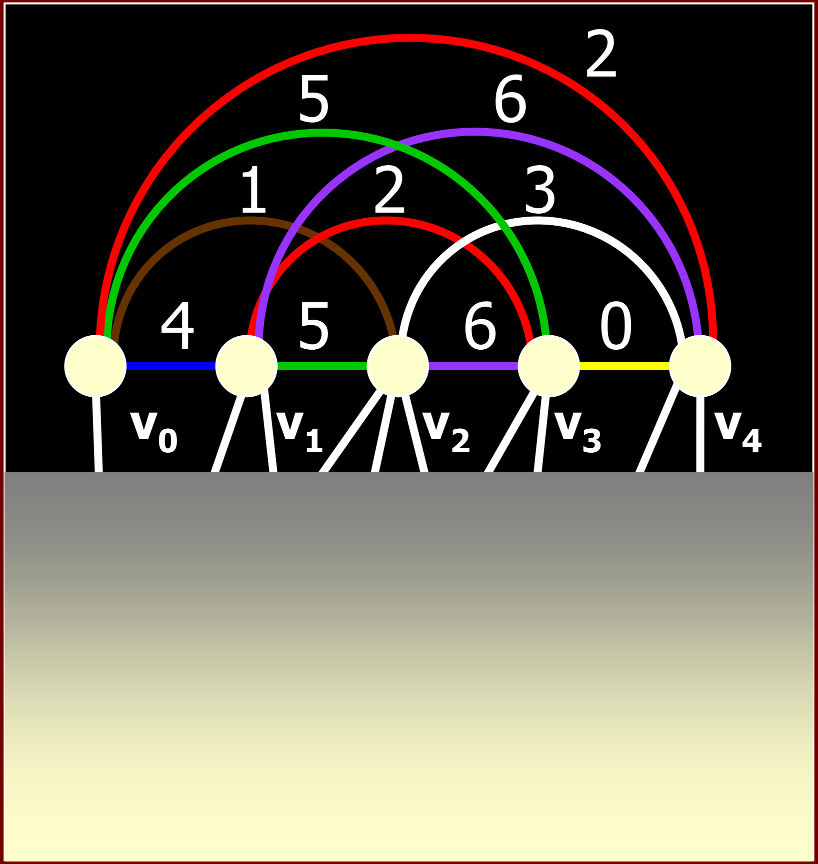
Using Latin Squares to Color Split Graphs

	0	1	2	3	4		
0	0	4	1	5	2	6	3
1	4	1	5	2	6	3	0
2	1	5	2	6	3	0	4
3	5	2	6	3	0	4	1
4	2	6	3	0	4	1	5
	6	3	0	4	1	5	2
	3	0	4	1	5	2	6

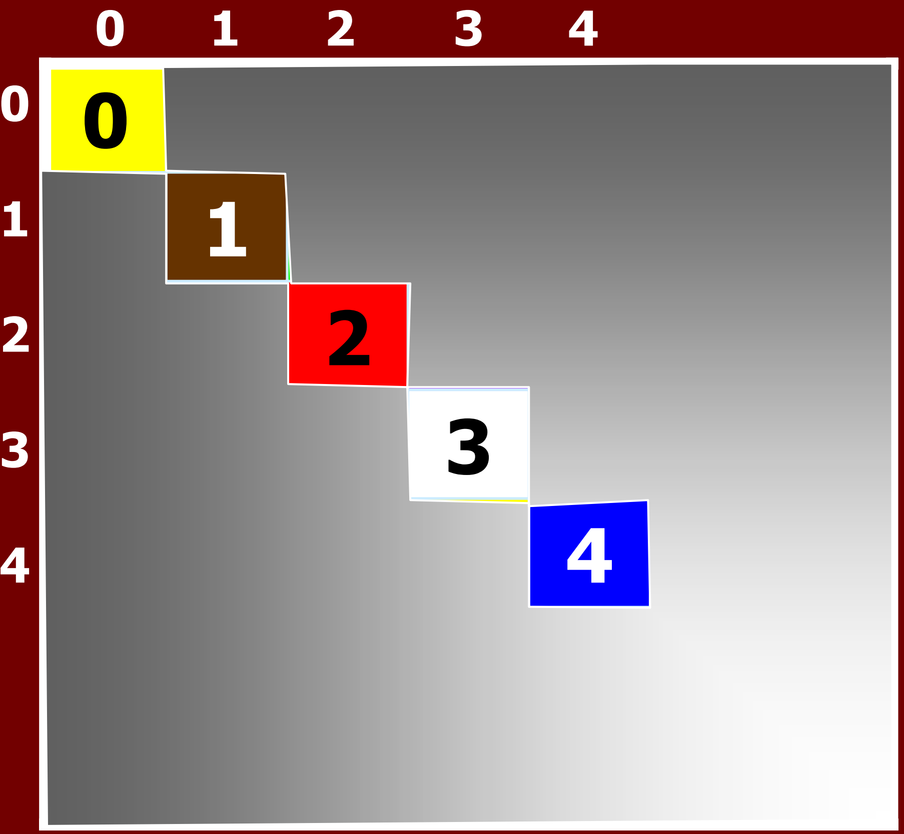


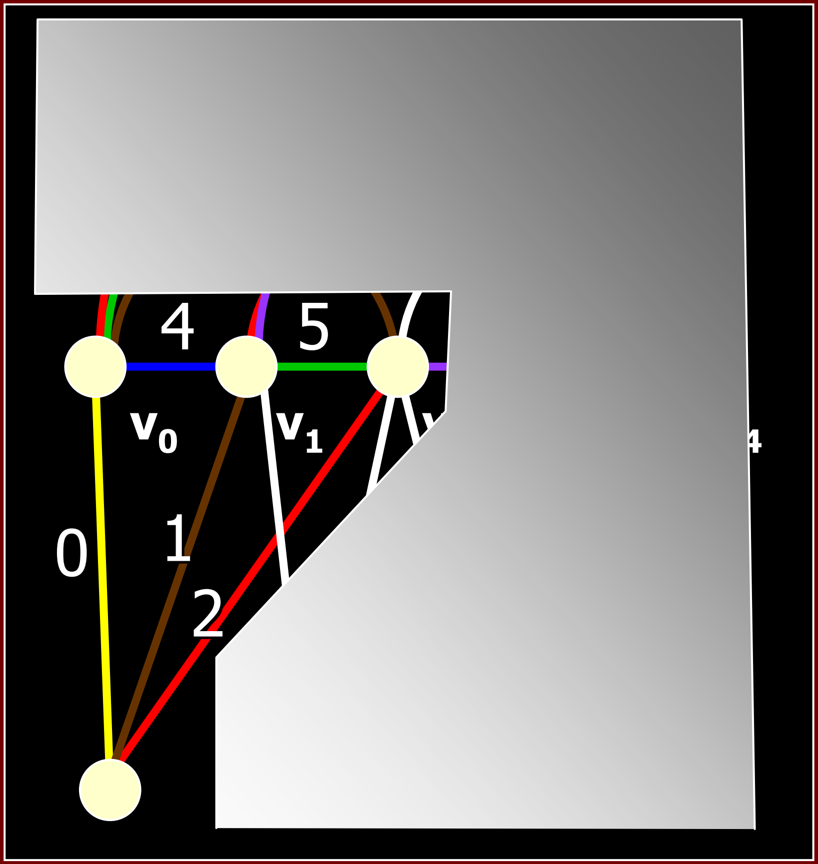
Using Latin Squares to Color Split Graphs

	0	1	2	3	4
0	0	4	1	5	2
1	4	1	5	2	6
2	1	5	2	6	3
3	5	2	6	3	0
4	2	6	3	0	4

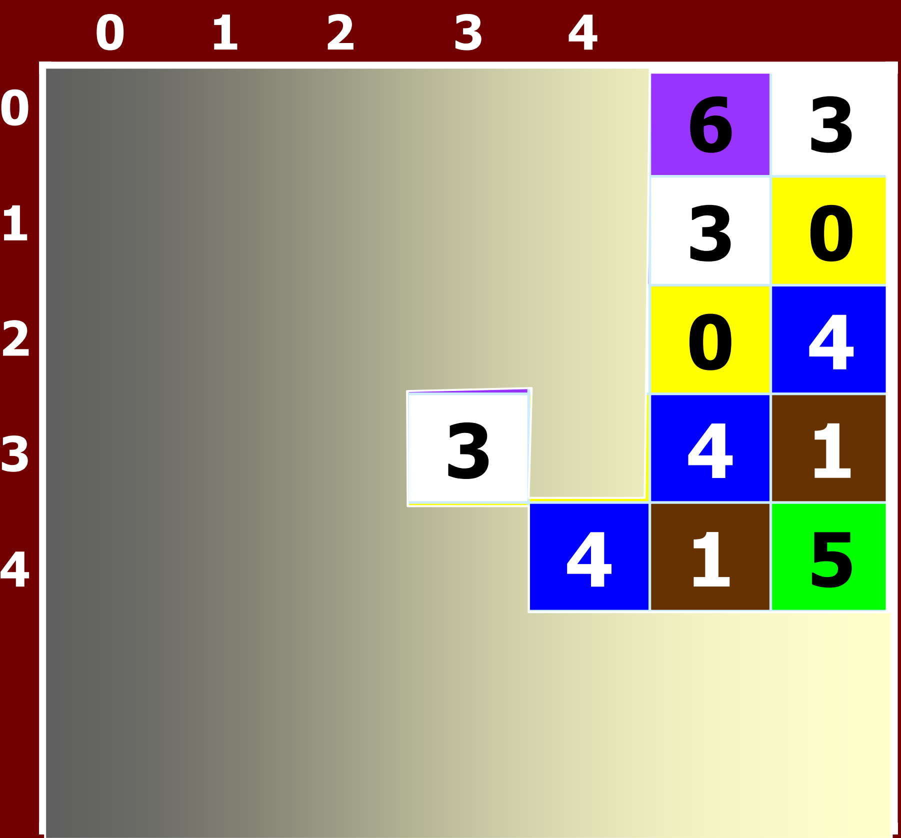


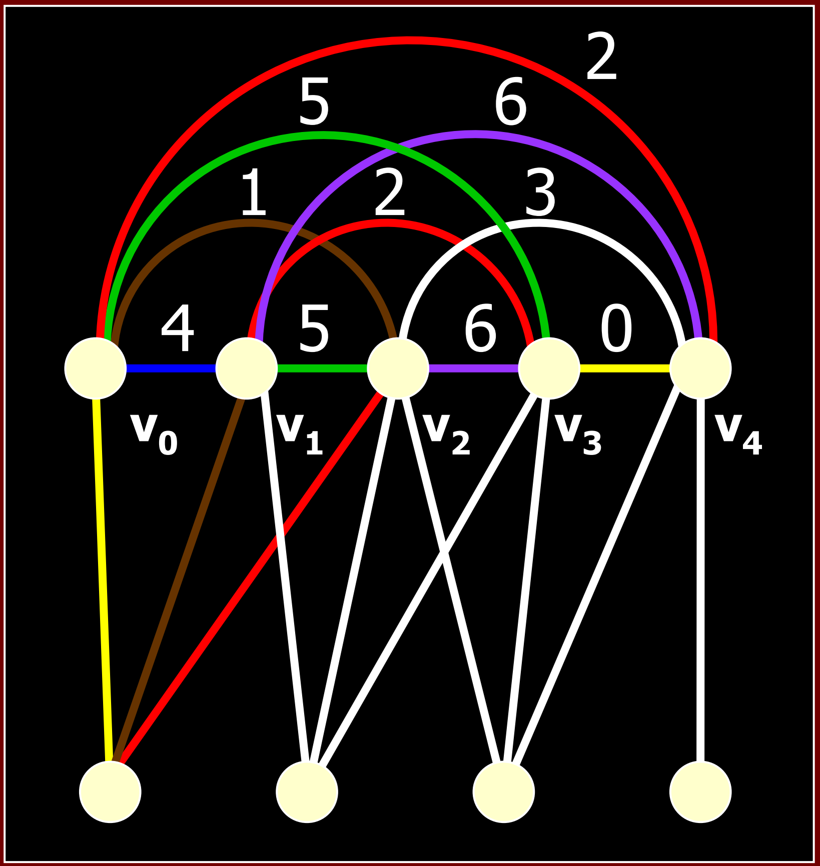
Using Latin Squares to Color Split Graphs



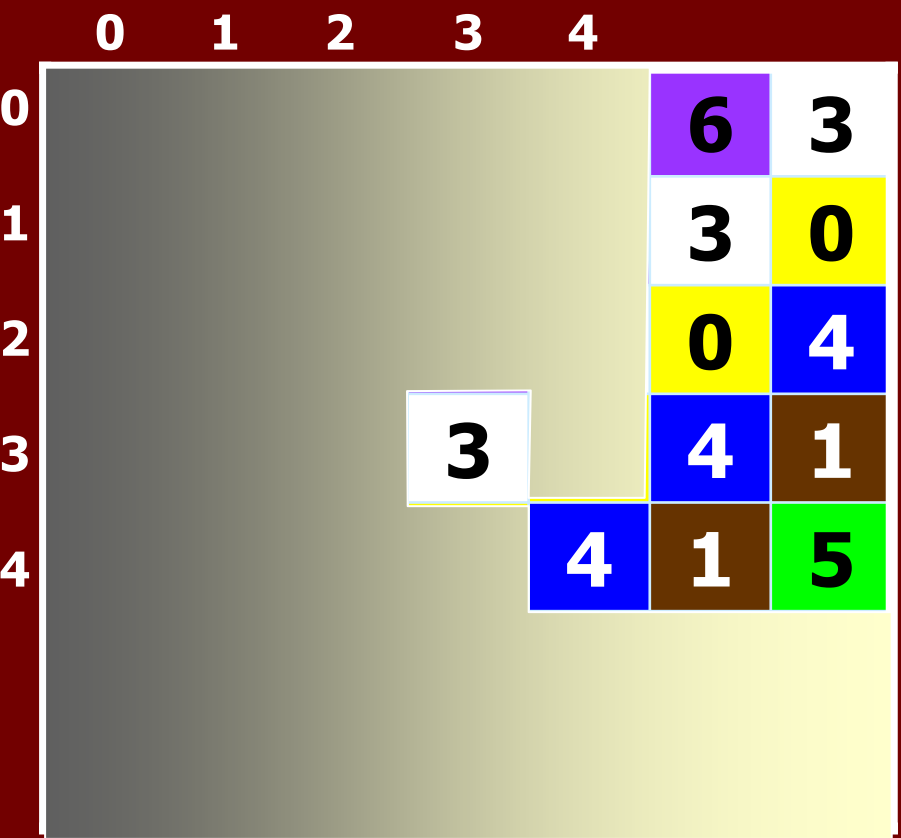


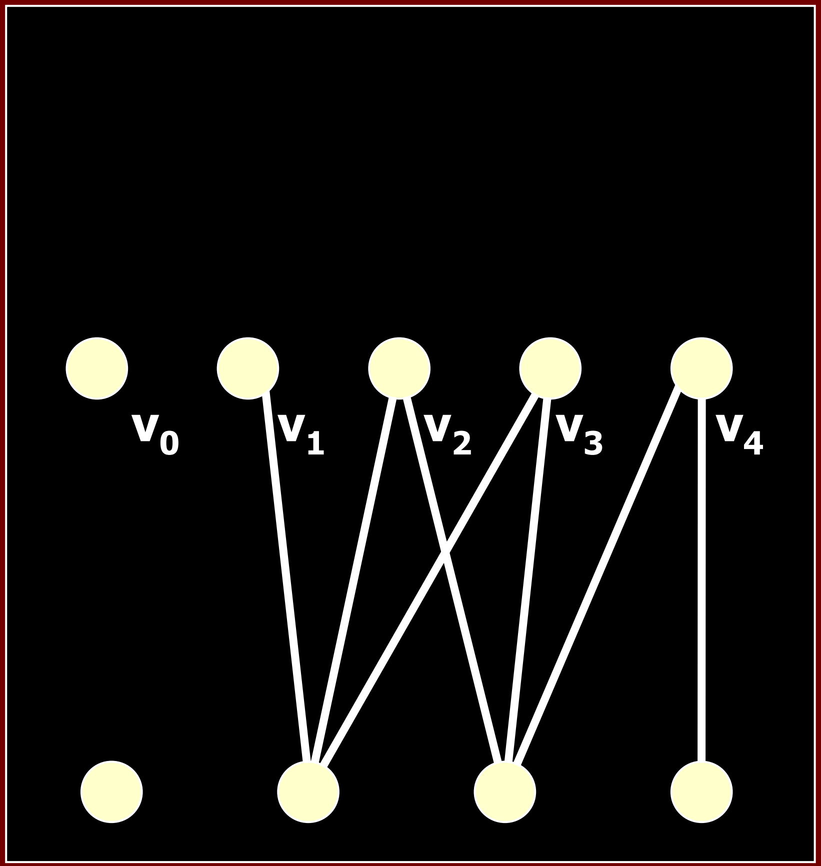
Using Latin Squares to Color Split Graphs





Using Latin Squares to Color Split Graphs





Using Latin Squares to Color Split Graphs

Let d_i be the number of elements in an array C_i.

Let $c_{i,j,}$ be the j^{th} entrie of the array $C_{i.}$

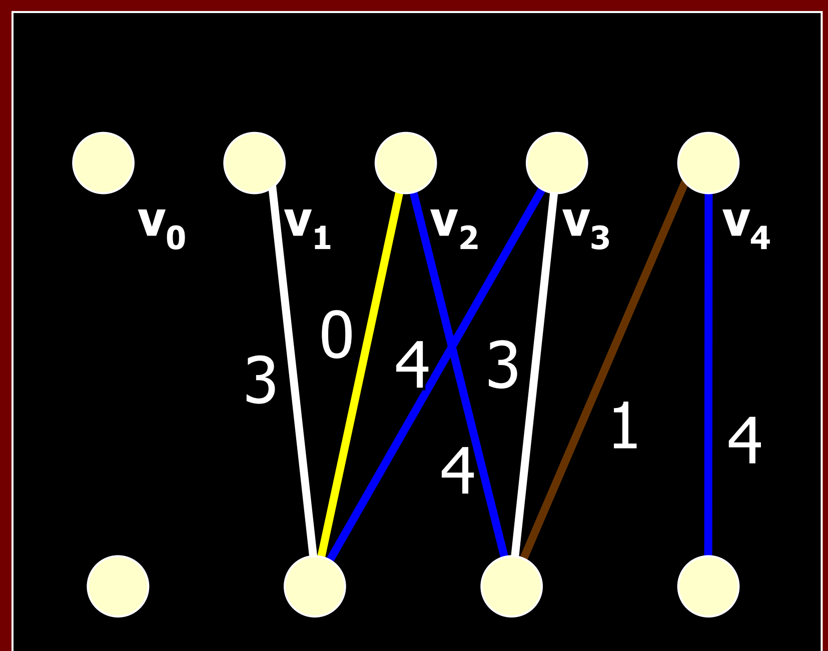
A set of arrays $\{C_0, \ldots, C_k\}$ is a ***monotonic color diagram*** if $c_{i,j,}$ occurs at most d_i-j times in the arrays $\{C_0, C_1, \ldots, C_{i-1}\}$.

	1	2	3
C_0	6	3	
C_1	3	0	
C_2	0	4	
C_3	3	4	1
C_4	4	1	5

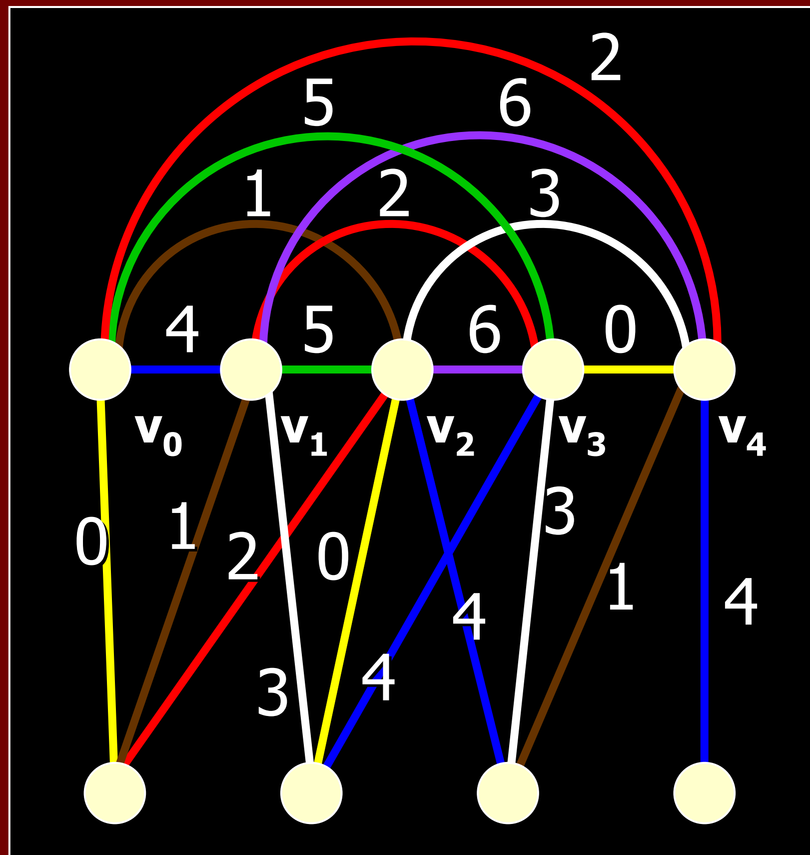
Using Latin Squares to Color Split Graphs

It is always possible to color a bipartite graph [Q,S] with a monotonic color diagram $\{C_0, C_1, \ldots C_{|Q|}\}$ if C_i has size at least $d(v_i)$.

	0	1	2
C_0	6	3	
C_1	3	0	
C_2	0	4	
C_3	3	4	1
C_4	4	1	5



Using Latin Squares to Color Split Graphs



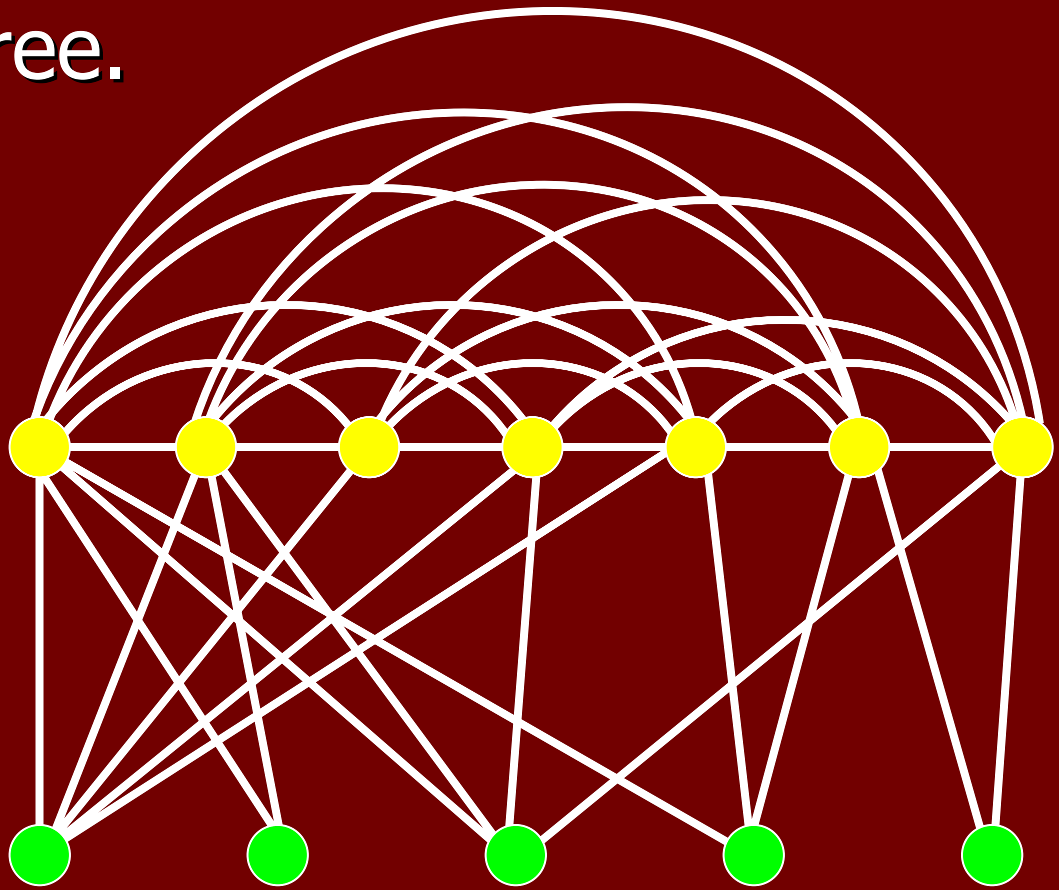
Using Latin Squares to Color Split Graphs

It is not possible to construct an idempotente commutative latin square of even order.

0	4	1	3	2	5
4	1	3	2	5	0
1	3	2	5	0	4
3	2	5	0	4	1
2	5	0	4	1	3
5	0	4	1	3	2

Using Latin Squares to Color Split Graphs

Let G be a split graph with even maximum degree.



$\Delta(G)=10$

Using Latin Squares to Color Split Graphs

Construct a commutative latin square of order $\Delta(G)-1$, where $m_{i,j} = i+j \pmod{\Delta(G)-1}$

	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8	0
2	2	3	4	5	6	7	8	0	1
3	3	4	5	6	7	8	0	1	2
4	4	5	6	7	8	0	1	2	3
5	5	6	7	8	0	1	2	3	4
6	6	7	8	0	1	2	3	4	5
7	7	8	0	1	2	3	4	5	6
8	8	0	1	2	3	4	5	6	7

Using Latin Squares to Color Split Graphs

Let's use this commutative latin square to construct:

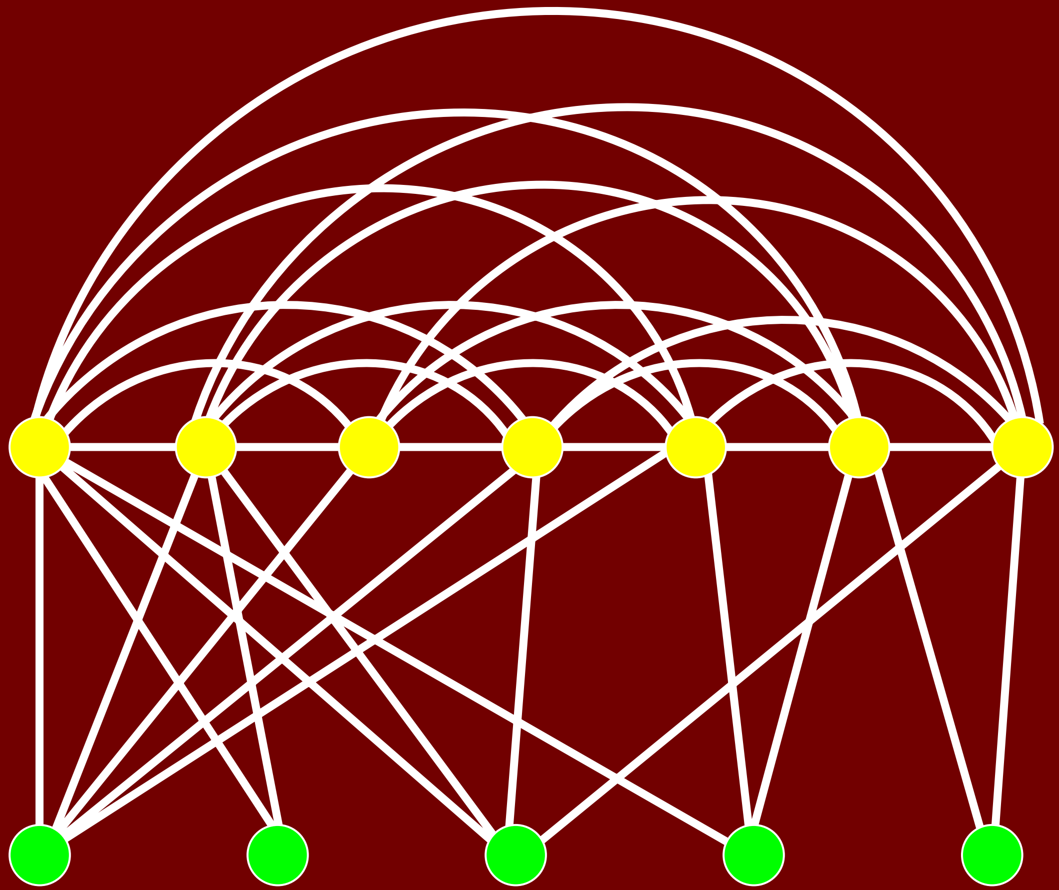
- a matrix A (that we use to color G[Q])

and

- a monotonic color diagram D (that we use to color a bipartite graph B=[Q,S]).

	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8	0
2	2	3	4	5	6	7	8	0	1
3	3	4	5	6	7	8	0	1	2
4	4	5	6	7	8	0	1	2	3
5	5	6	7	8	0	1	2	3	4
6	6	7	8	0	1	2	3	4	5
7	7	8	0	1	2	3	4	5	6
8	8	0	1	2	3	4	5	6	7

Using Latin Squares to Color Split Graphs



|Q|=7

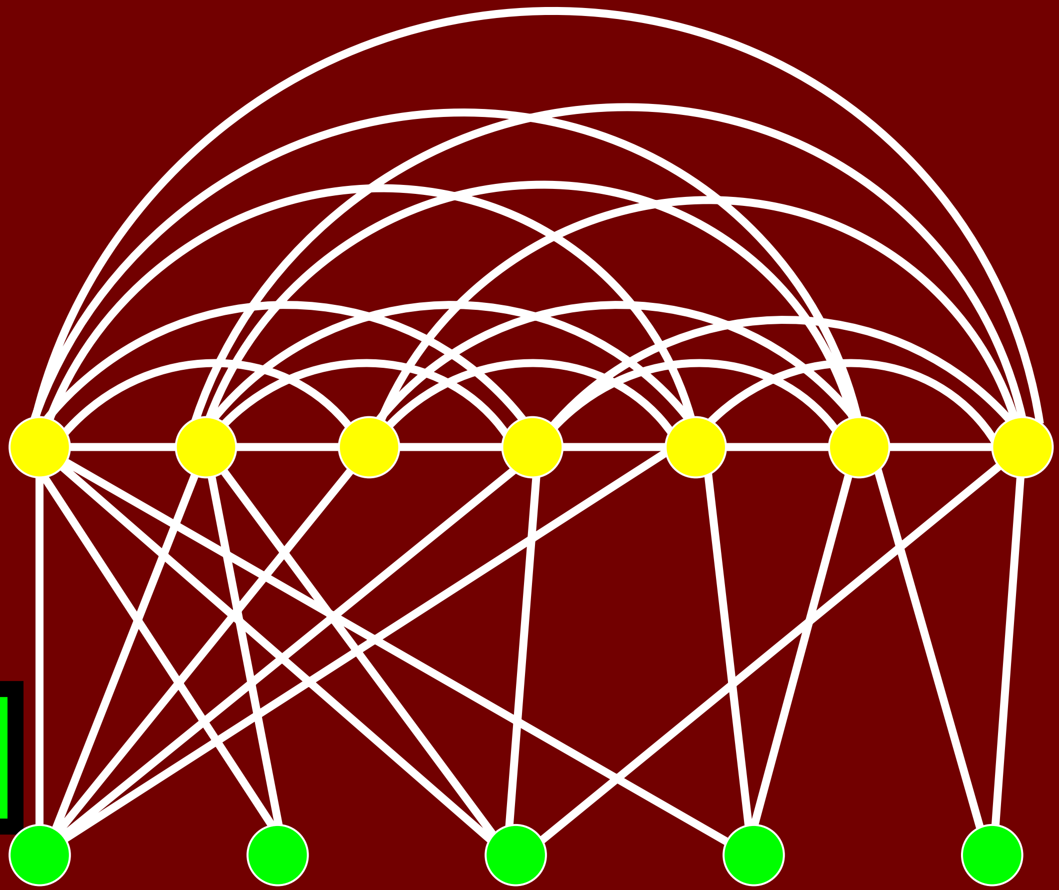
Using Latin Squares to Color Split Graphs

	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8	0
2	2	3	4	5	6	7	8	0	1
3	3	4	5	6	7	8	0	1	2
4	4	5	6	7	8	0	1	2	3
5	5	6	7	8	0	1	2	3	4
6	6	7	8	0	1	2	3	4	5
7	7	8	0	1	2	3	4	5	6
8	8	0	1	2	3	4	5	6	7

Using Latin Squares to Color Split Graphs

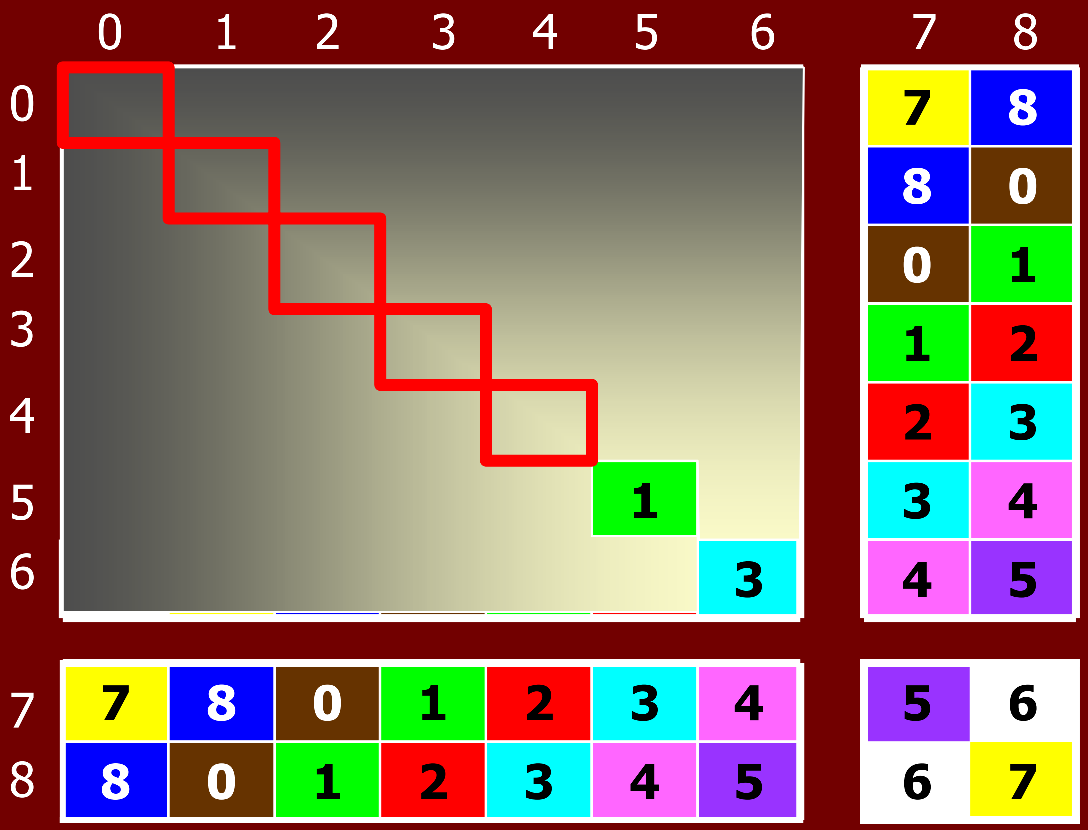
	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8	0
2	2	3	4	5	6	7	8	0	1
3	3	4	5	6	7	8	0	1	2
4	4	5	6	7	8	0	1	2	3
5	5	6	7	8	0	1	2	3	4
6	6	7	8	0	1	2	3	4	5
7	7	8	0	1	2	3	4	5	6
8	8	0	1	2	3	4	5	6	7

Using Latin Squares to Color Split Graphs

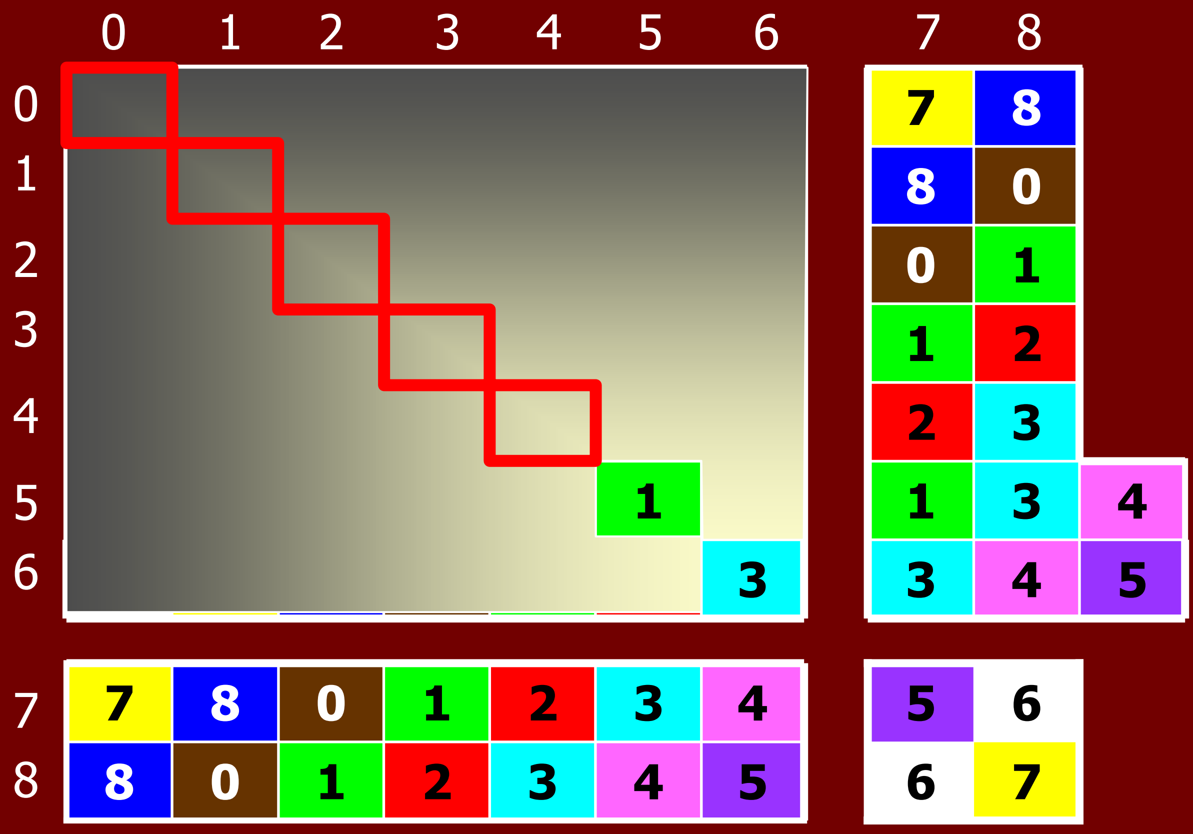


d(S)=5

Using Latin Squares to Color Split Graphs



Using Latin Squares to Color Split Graphs



Using Latin Squares to Color Split Graphs

	0	1	2	3	4	5	6	7	8		
0	0	1	2	3	4	5	6	9	7	8	
1	1	2	3	4	5	6	7	9	8	0	
2	2	3	4	5	6	7	8	9	0	1	
3	3	4	5	6	7	8	0	9	1	2	
4	4	5	6	7	8	0	1	9	2	3	
5	5	6	7	8	0	1	2	9	1	3	4
6	6	7	8	0	1	2	3	9	3	4	5
7	7	8	0	1	2	3	4	5	6		
8	8	0	1	2	3	4	5	6	7		

Using Latin Squares to Color Split Graphs

	0	1	2	3	4	5	6	7	8		
0	0	1	2	3	4	5	6	9	7	8	
1	1	2	3	4	5	6	7	9	8	0	
2	2	3	4	5	6	7	8	9	0	1	
3	3	4	5	6	7	8	0	9	1	2	
4	4	5	6	7	8	0	1	9	2	3	
5	5	6	7	8	0	1	2	9	1	3	4
6	6	7	8	0	1	2	3	9	3	4	5
7	7	8	0	1	2	3	4	5	6		
8	8	0	1	2	3	4	5	6	7		

Using Latin Squares to Color Split Graphs

	0	1	2	3	4	5	6	7	8		
0	0	1	2	3	4	5	9	6	7	8	
1	1	2	3	4	5	6	7	9	8	0	
2	2	3	4	5	6	7	8	9	0	1	
3	3	4	5	6	7	8	0	9	1	2	
4	4	5	6	7	8	0	1	9	2	3	
5	5	6	7	8	0	1	2	9	1	3	4
6	9	7	8	0	1	2	3	6	3	4	5
7	7	8	0	1	2	3	4	5	6		
8	8	0	1	2	3	4	5	6	7		

Using Latin Squares to Color Split Graphs

	0	1	2	3	4	5	6	7	8		
0	0	1	2	3	4	5	9	6	7	8	
1	1	2	3	4	5	9	7	6	8	0	
2	2	3	4	5	6	7	8	9	0	1	
3	3	4	5	6	7	8	0	9	1	2	
4	4	5	6	7	8	0	1	9	2	3	
5	5	9	7	8	0	1	2	6	1	3	4
6	9	7	8	0	1	2	3	6	3	4	5
7	7	8	0	1	2	3	4	5	6		
8	8	0	1	2	3	4	5	6	7		

Using Latin Squares to Color Split Graphs

	0	1	2	3	4	5	6	7	8		
0	0	1	2	3	4	5	9	6	7	8	
1	1	2	3	4	5	9	7	6	8	0	
2	2	3	4	5	6	7	8	9	0	1	
3	3	4	5	6	7	8	0	9	1	2	
4	4	5	6	7	8	0	1	9	2	3	
5	5	9	7	8	0	1	2	6	1	3	4
6	9	7	8	0	1	2	3	6	3	4	5
7	7	8	0	1	2	3	4	5	6		
8	8	0	1	2	3	4	5	6	7		

Using Latin Squares to Color Split Graphs

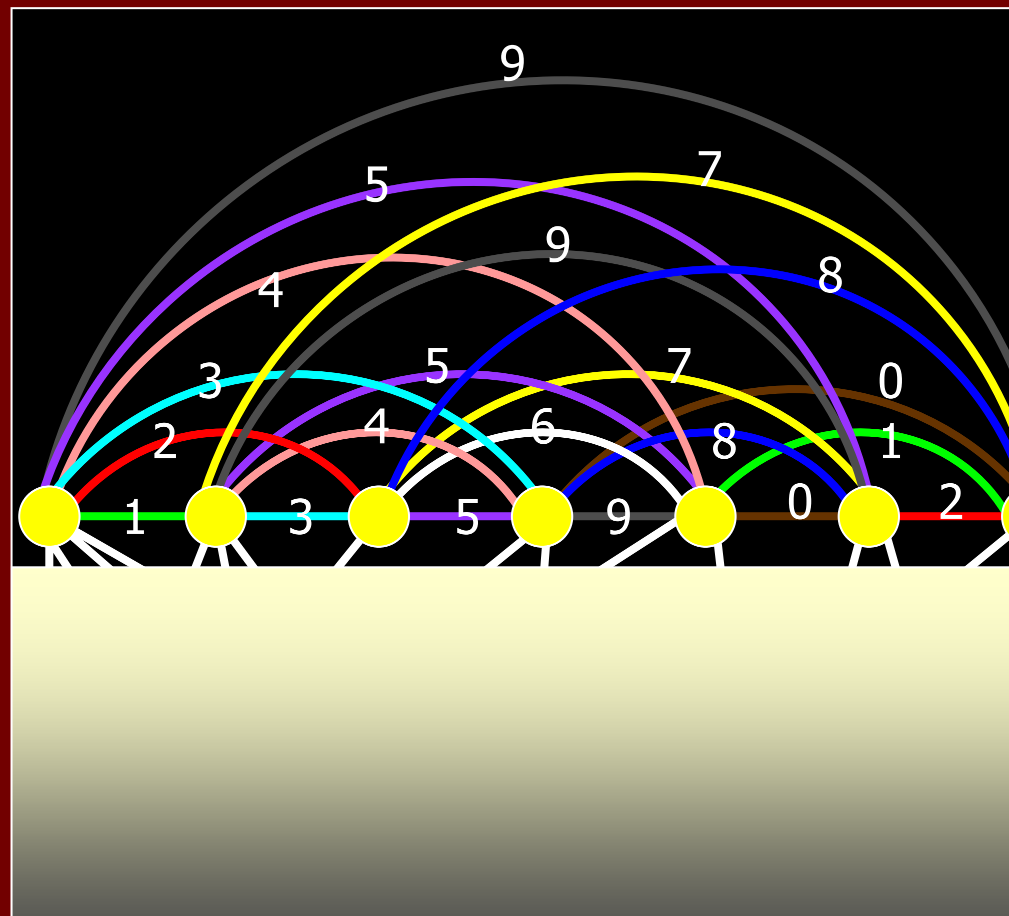
	0	1	2	3	4	5	6	7	8		
0	0	1	2	3	4	5	9	6	7	8	
1	1	2	3	4	5	9	7	6	8	0	
2	2	3	4	5	6	7	8	9	0	1	
3	3	4	5	6	7	8	0	9	1	2	
4	4	5	6	7	8	0	1	9	2	3	
5	5	9	7	8	0	1	2	6	1	3	4
6	9	7	8	0	1	2	3	6	3	4	5
7	7	8	0	1	2	3	4	5	6		
8	8	0	1	2	3	4	5	6	7		

Using Latin Squares to Color Split Graphs

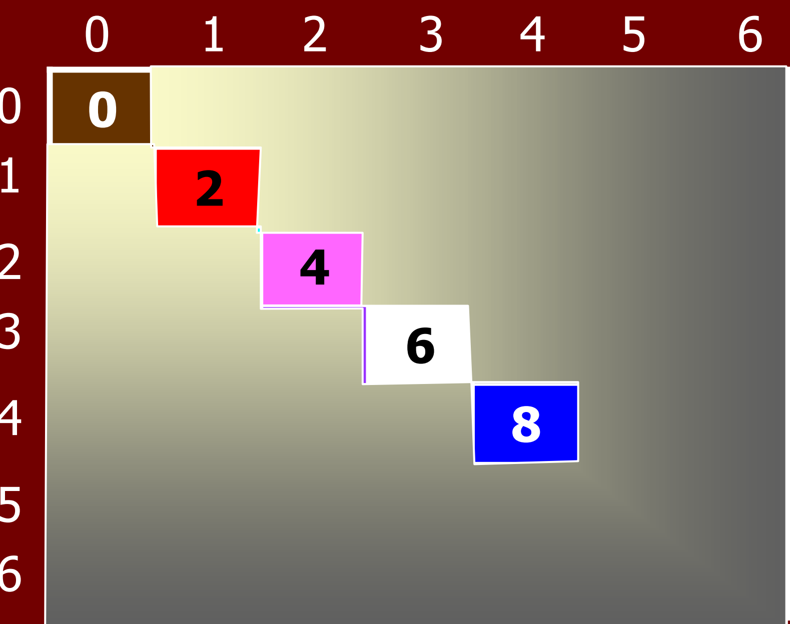
	0	1	2	3	4	5	6	7	8		
0	0	1	2	3	4	5	9	6	7	8	
1	1	2	3	4	5	9	7	6	8	0	
2	2	3	4	5	6	7	8	9	0	1	
3	3	4	5	6	9	8	0	7	1	2	
4	4	5	6	9	8	0	1	7	2	3	
5	5	9	7	8	0	1	2	6	1	3	4
6	9	7	8	0	1	2	3	6	3	4	5
7	7	8	0	1	2	3	4	5	6		
8	8	0	1	2	3	4	5	6	7		

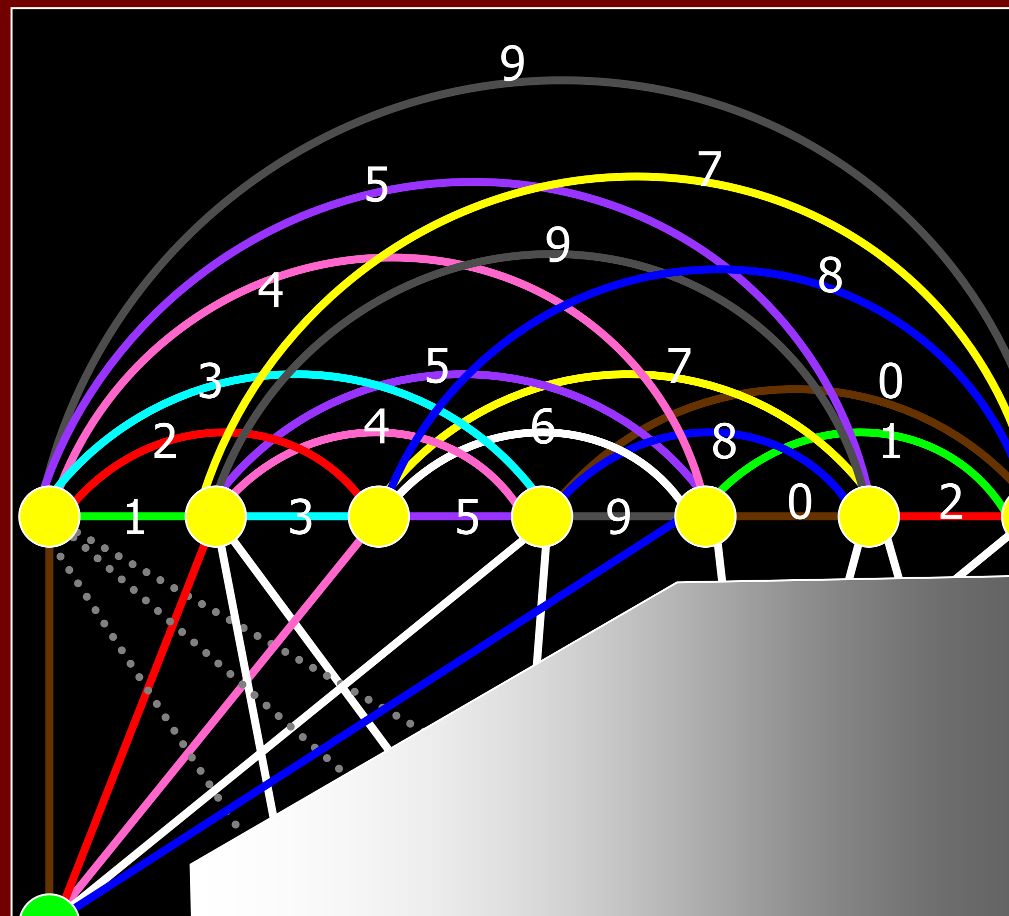
Using Latin Squares to Color Split Graphs

	0	1	2	3	4	5	6
0	**0**	**1**	**2**	**3**	**4**	**5**	**9**
1	**1**	**2**	**3**	**4**	**5**	**9**	**7**
2	**2**	**3**	**4**	**5**	**6**	**7**	**8**
3	**3**	**4**	**5**	**6**	**9**	**8**	**0**
4	**4**	**5**	**6**	**9**	**8**	**0**	**1**
5	**5**	**9**	**7**	**8**	**0**	**1**	**2**
6	**9**	**7**	**8**	**0**	**1**	**2**	**3**



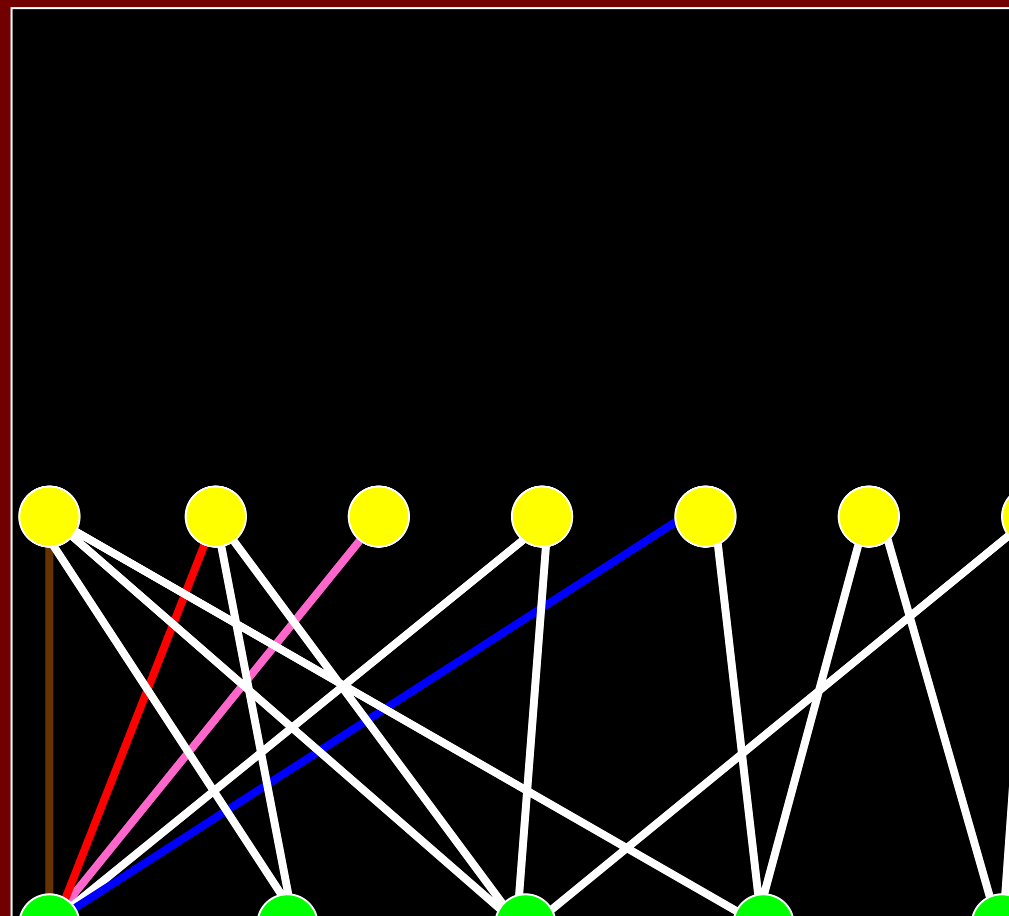
Using Latin Squares to Color Split Graphs





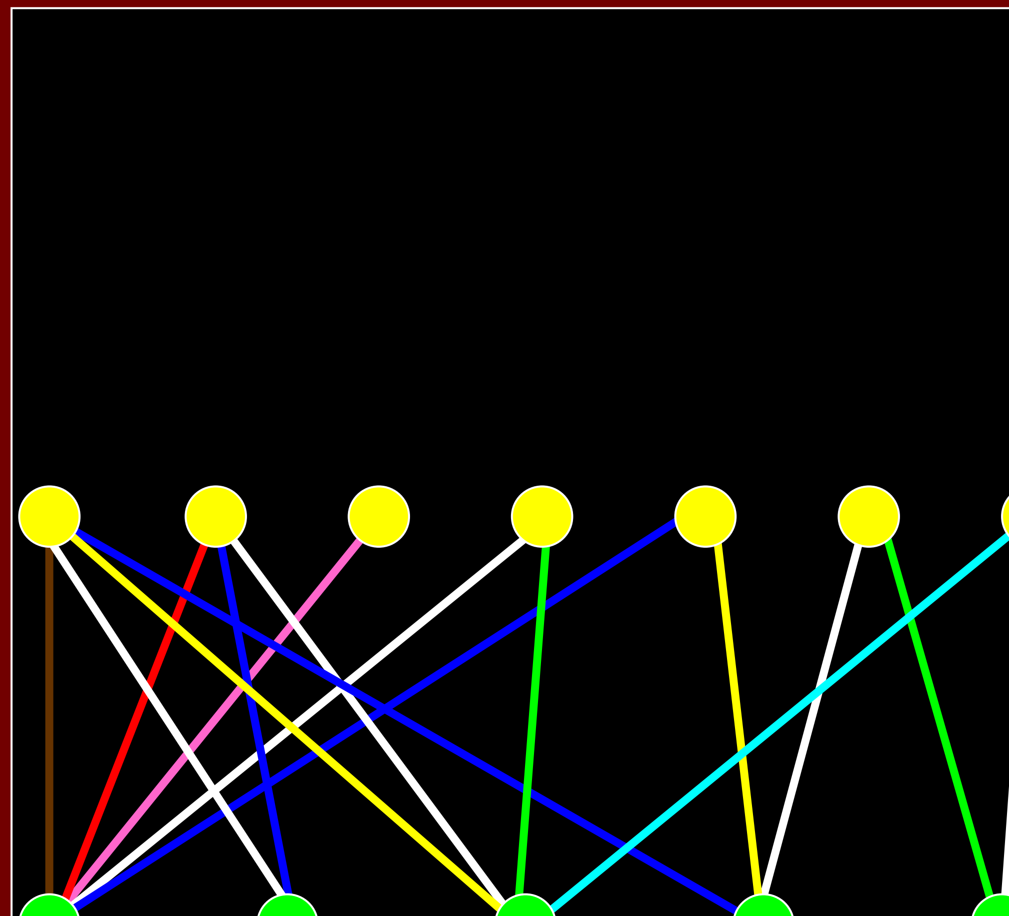
Using Latin Squares to Color Split Graphs

	1	2	4	5
0	6	7	8	
1	6	8	0	
2	9	0	1	
3	7	1	2	
4	7	8	2	3
5	6	1	3	4
6	6	3	4	5

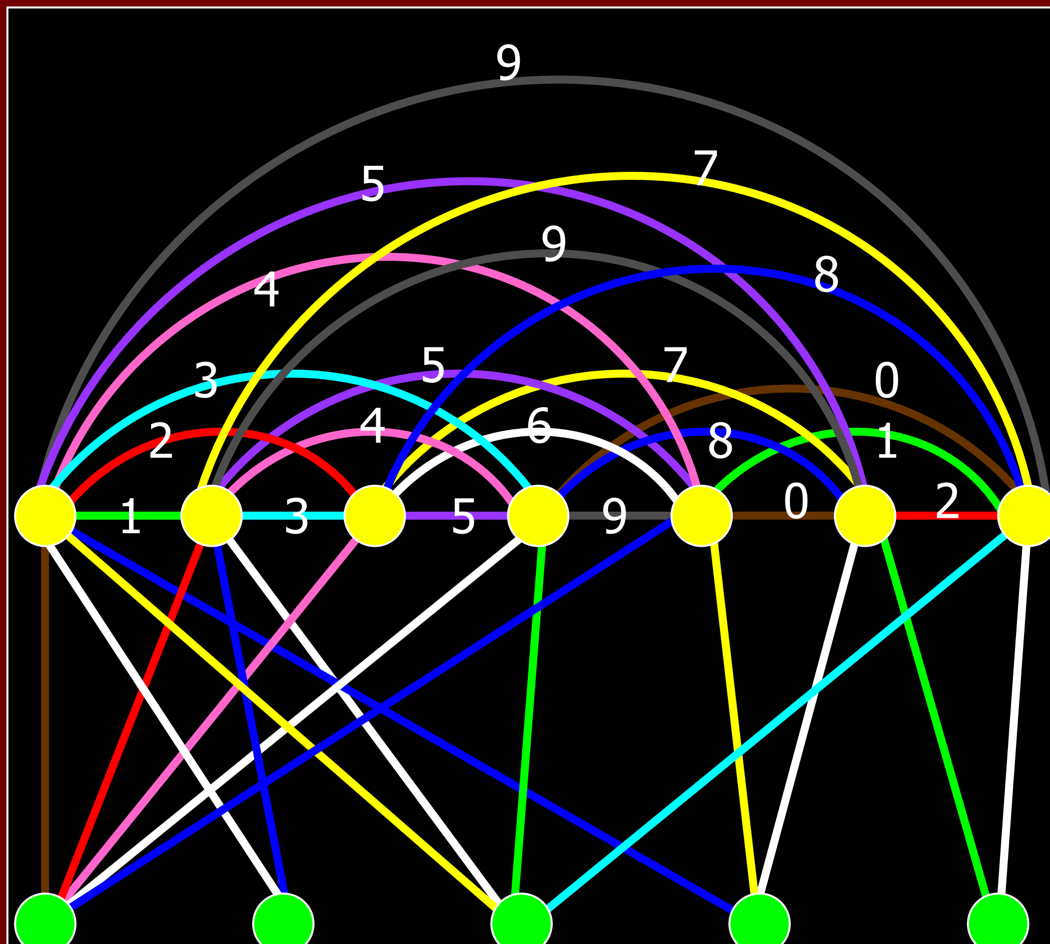


Using Latin Squares to Color Split Graphs

	1	2	4	5
0	6	7	8	
1	6	8	0	
2	9	0	1	
3	7	1	2	
4	7	8	2	3
5	6	1	3	4
6	6	3	4	5



Using Latin Squares to Color Split Graphs



Using Latin Squares to Color Split Graphs

	0	1	2	3	4	5	6	7	8		
0	0	1	2	3	4	5	9	6	7	8	
1	1	2	3	4	5	9	7	6	8	0	
2	2	3	4	5	6	7	8	9	0	1	
3	3	4	5	6	9	8	0	7	1	2	
4	4	5	6	9	8	0	1	7	2	3	
5	5	9	7	8	0	1	2	6	1	3	4
6	9	7	8	0	1	2	3	6	3	4	5
7	7	8	0	1	2	3	4	5	6		
8	8	0	1	2	3	4	5	6	7		

Color of the element $m_{(|Q|+1)/2,(|Q|+1)/}$

Using Latin Squares to Color Split Graphs

	0	1	2	3	4	5	6	7	8	
0	0	1	2	3	4	5	9	6	7	8
1	1	2	3	4	5	9	7	6	8	0
2	2	3	4	5	6	7	8	[illegible]	0	1
3	3	4	5	6	9	8	0	7	6	1	2
4	4	5	6	9	8	0	1	7	8	2	3
5	5	9	7	8	0	1	2	6	1	3	4
6	9	7	8	0	1	2	3	6	3	4	5
7	7	8	0	1	2	3	4	5	6	
8	8	0	1	2	3	4	5	6	7	

Using Latin Squares to Color Split Graphs

	0	1	2	3	4	5	6	7	8		
0	0	1	2	3	4	5	9	6	7	8	
1	1	2	3	4	5	9	7	6	8	0	
2	2	3	4	5	6	7	8	[illegible]	0	1	
3	3	4	5	6	9	8	0	7	6	1	2
4	4	5	6	9	8	0	1	7	8	2	3
5	5	9	7	8	0	1	2	6	1	3	4
6	9	7	8	0	1	2	3	6	3	4	5
7	7	8	0	1	2	3	4	5	6		
8	8	0	1	2	3	4	5	6	7		

Necessary condition: there is a vertex in S with degree at least |Q|/2

Using Latin Squares to Color Split Graphs

Number of times that the color (Δ(G)-1) can appears in the monotonic color diagram is at most d(Q)-1, so:

$$|Q|-(\Delta-|Q|+2(\Delta-|Q|-2)+(\Delta-|Q|-4)+ \dots +(\Delta-|Q|-(\Delta-|Q|-2)) =$$

$$|Q|-(d(Q)-1+2(d(Q)-3+ \dots +d(Q)-(d(Q)-2))) =$$

$$|Q| - (d(Q) -1 + ((d(Q)-1) (d(Q)-3))/2 = |Q| - ((d(Q)-1)^2)/2$$

$$|Q| - ((d(Q)-1)^2)/2 \le d(Q)-1 \rightarrow 2|Q| - (d(Q)-1)^2 \le 2d(Q)-2$$

$$\rightarrow 2|Q| \le 2d(Q)-2 + (d(Q)-1)^2 \rightarrow |Q| \le (d(Q))^2-1 \rightarrow (d(Q))^2 \ge 2|Q|+1$$

Necessary condition: $(d(Q))^2 \ge 2|Q|+1$

The Classification Problem for Split Graphs with even maximum degree

Let G be a split graph with even maximum degree. If G has a vertex in S with degree at least $|Q|/2$ and $d(Q)^2 \geq 2|Q|+1$, then G is Class 1.

Thank you