A decomposition for total-coloring graphs of maximum degree 3

Cologne-Twente Workshop 2008

Raphael Machado and

Celina M. H. de Figueiredo

Gargnano. 14/05/2008.

General concepts

- We work with simple graphs $G=(V(G),E(G))$
  - $V(G)$ is the set of *vertices*
  - $E(G)$ is the set of *edges* (unordered pairs of vertices)

- Set of *elements* of G: $S(G)=V(G)\cup E(G)$

General concepts

- Vertices $u, v \in V(G)$ are *adjacent* if $uv \in E(G)$
- Edges $e_1, e_2 \in E(G)$ are *adjacent* if they have a common endvertex
- Vertex $u \in V(G)$ and edge $e \in E(G)$ are *incident* if u is endvertex of e

Some notation

- Open neighborhood: $Adj_G(u) := \{v \in V(G) | uv \in E(G)\}$
- Closed neighborhood: $N_G(u) := Adj_G(u) \cup \{u\}$
- Degree: $deg_G(u) := |Adj_G(u)|$
- Maximum degree: $\Delta(G)$

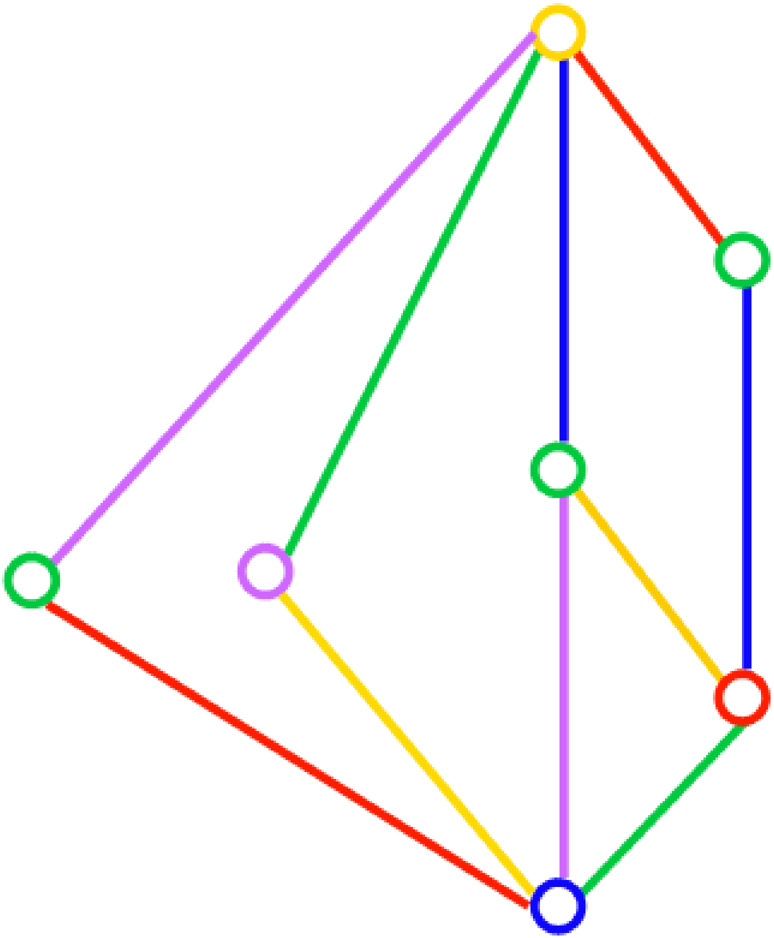
Graph colorings

- Associated to conflict models
- Related elements (incident or adjacent) receive distinct colors
- Three classical models
  - Vertex-coloring
  - Edge-coloring
  - Total-coloring

Total-coloring

- Is an association of colors to the elements of a graph
- Incident or adjacent elements receive distinct colors
- *k*-total-coloring: a total-coloring that uses *k* colors
- *k*-total-colorable graph: it can be colored with *k* colors
- Total chromatic number $\chi_T(G)$: least number of colors sufficient for total-coloring G

Example of 5-total-coloring

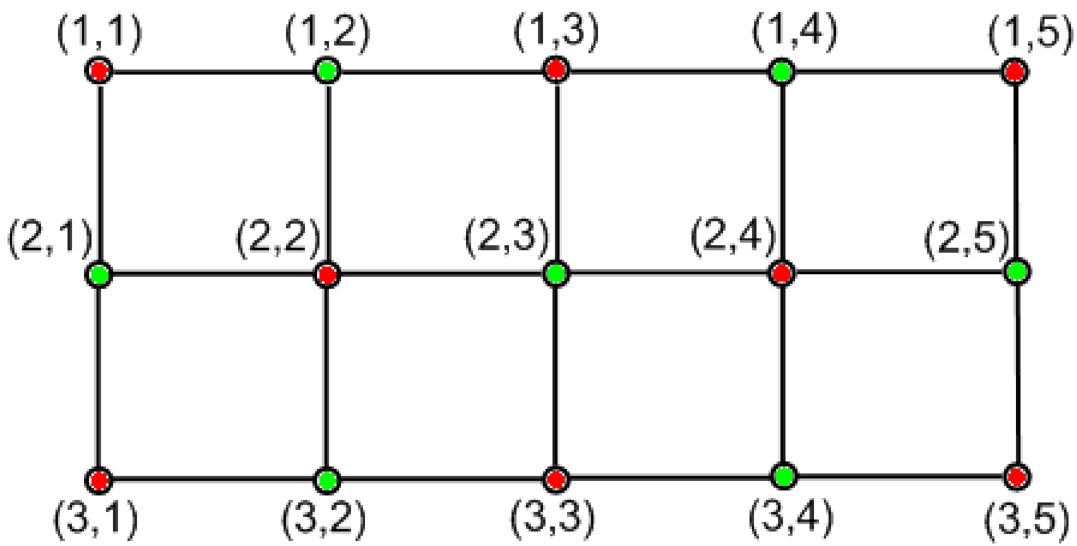


Some definitions and results

- Observe that $\chi_T(G) \geq \Delta(G) + 1$
- Total Coloring Conjecture: $\chi_T(G) \leq \Delta(G) + 2$
- Classification problem:
  - A graph is <u>Type 1</u> if $\chi_T(G) = \Delta(G) + 1$
  - A graph is <u>Type 2</u> if $\chi_T(G) = \Delta(G) + 2$
- It is NP-complete to determine if a graph is Type 1
  - It remais NP-complete even for cubic bipartite graphs.

Grids

- G_{mxn} is a grid if
  - $V(G_{mxn}) = \{1,\ldots,m\} \times \{1,\ldots,n\}$
  - $E(G_{mxn}) = \{(i,j)(s,t): |i-s|+|j-t|=1\}$

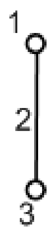
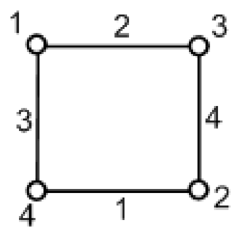
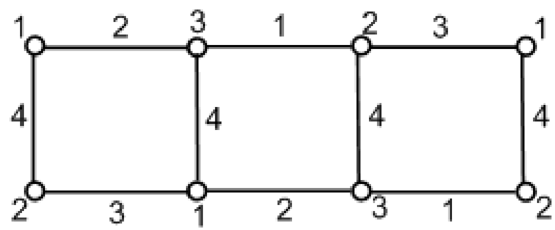
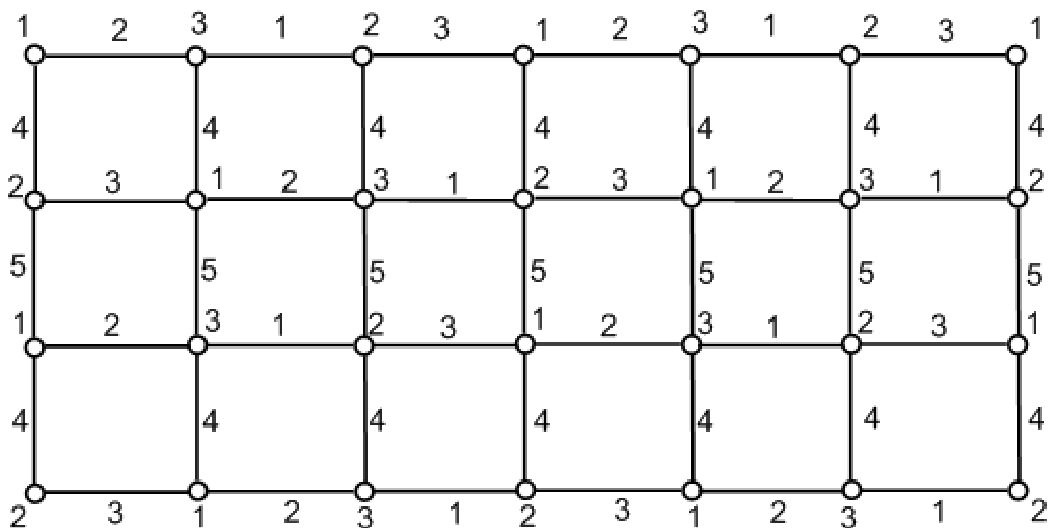


Planar

Bipartite

Total-coloring grids

- P_2 and C_4 are Type 2
- All other grids are Type 1

Partial-grids

- Arbitrary subgraphs of grids
  - Recognition: unknown complexity
- Total chromatic number determined for $\Delta = 0$, 1, 2 and 4
- Open problem for $\Delta=3$.
  - All known examples are Type 1
    - Trees
    - At most three maximum degree vertices
    - Maximum induced cycle 4

Our main results

- Development of a decomposition method for total coloring graphs of maximum degree 3.
- Classification of partial grids with maximum degree 3 and maximum induced cycle 8 as Type 1 (using the decomposition method developed)
- (Some recent results in series-parallel graphs total-coloring)

Decomposition for total-coloring: the biconnected components

- As a first step we formalize a result that allows us to focus on biconnected graphs.

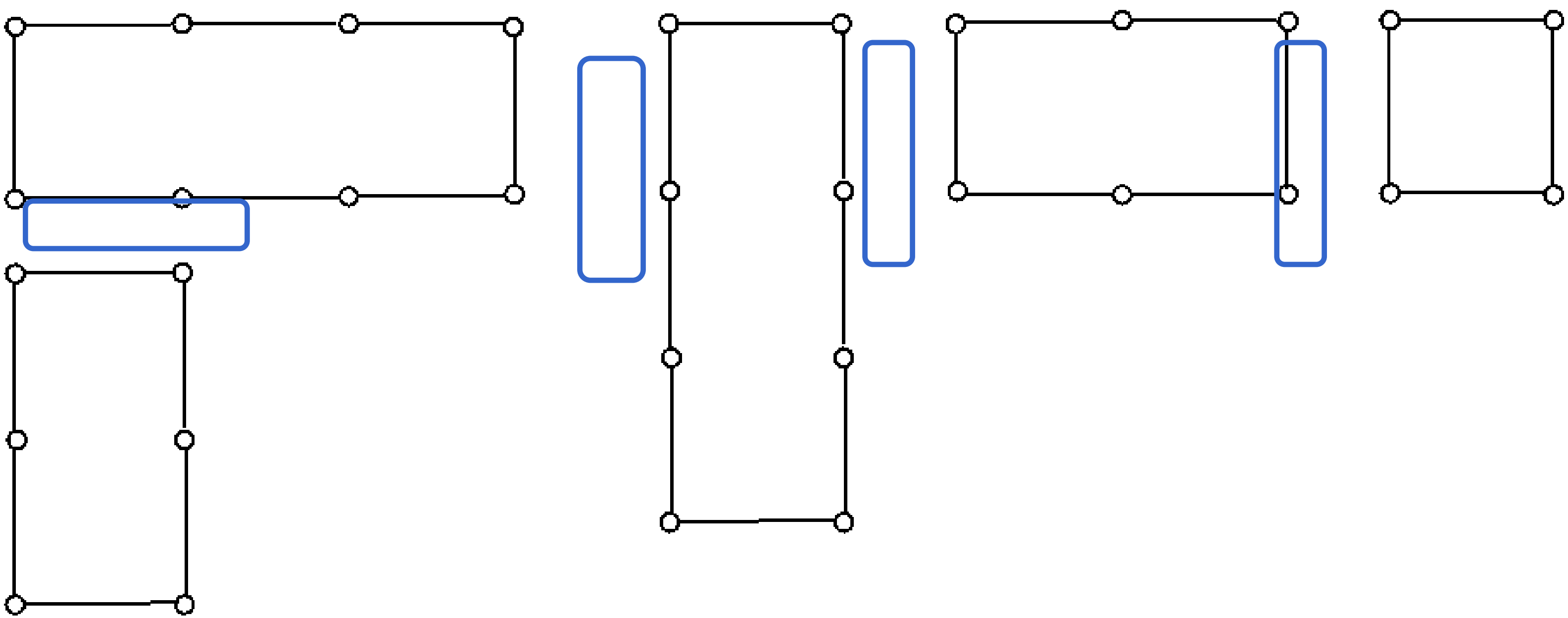
- If G is a graph such that all of its biconnected components have an α-total-coloring ($\alpha \geq \Delta(G) + 1$), then G itself has an α-total-coloring.

Decomposition for total-coloring: K_2-cut-free components

- A <u>cut</u> of a graph G is a set of vertices whose exclusion disconnets G.
- If $C \subseteq V(G)$ is a cut whose exclusion defines the components $G_1,\ldots,G_j$, the <u>C-components</u> of G are $G[V(G_1) \cup C],\ldots,G[V(G_j) \cup C]$
- A <u>K_2-cut</u> is a cut $\{u,v\}$ such that u and v are adjacent.

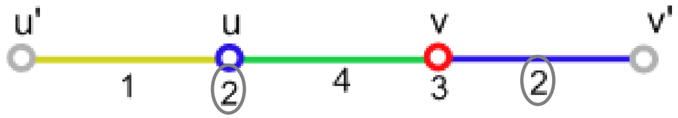
Decomposition for total-coloring: K_2-cut-free components

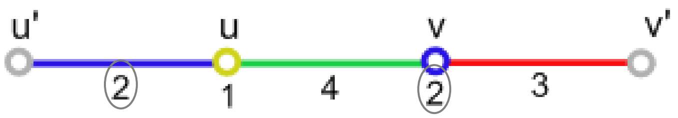
- The K_2-cut-free components of G are defined by the recursive application of K_2-cuts in this graph.



Decomposition for total-coloring: frontier-candidates

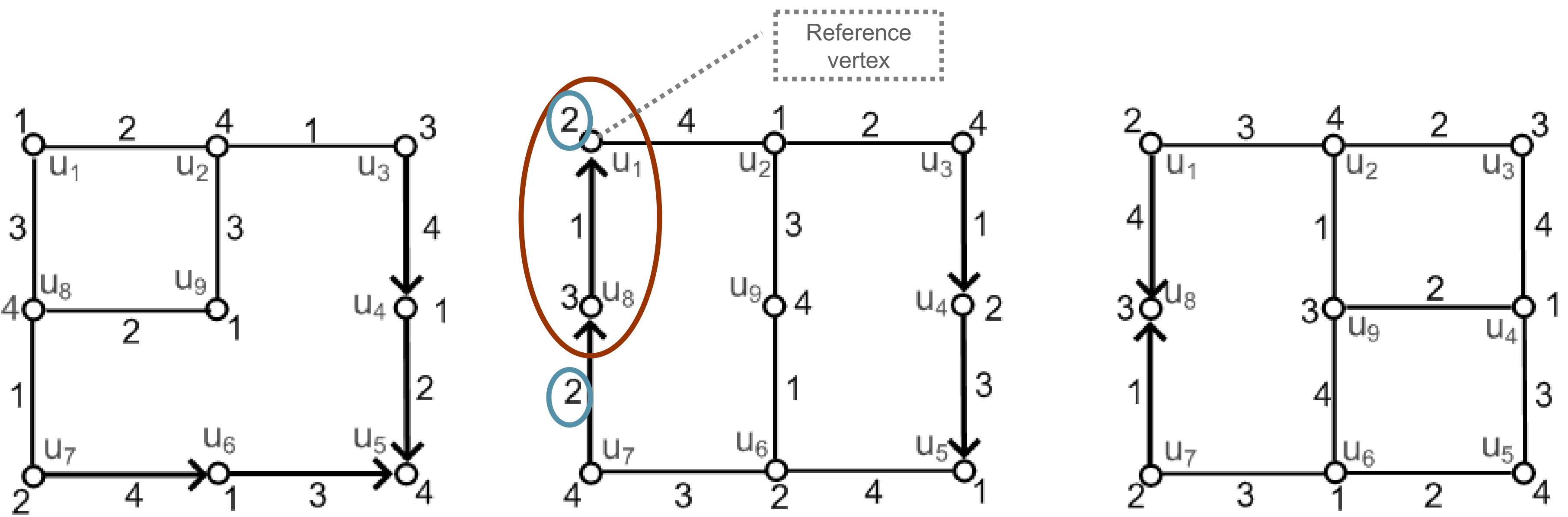
- The set {u,v} is a <u>frontier-candidate</u> if u and v are adjacent vertices and both have degree 2.
- Let {u,v} be a frontier candidate and denote $u' \neq v$ and $v' \neq u$ the neighbohrs, respectively, of u and v
- We say that a coloring <u>satisfies the frontier condition for {u,v}</u> if $u'u$, u, uv, v and vv' are colored in one of the following ways:



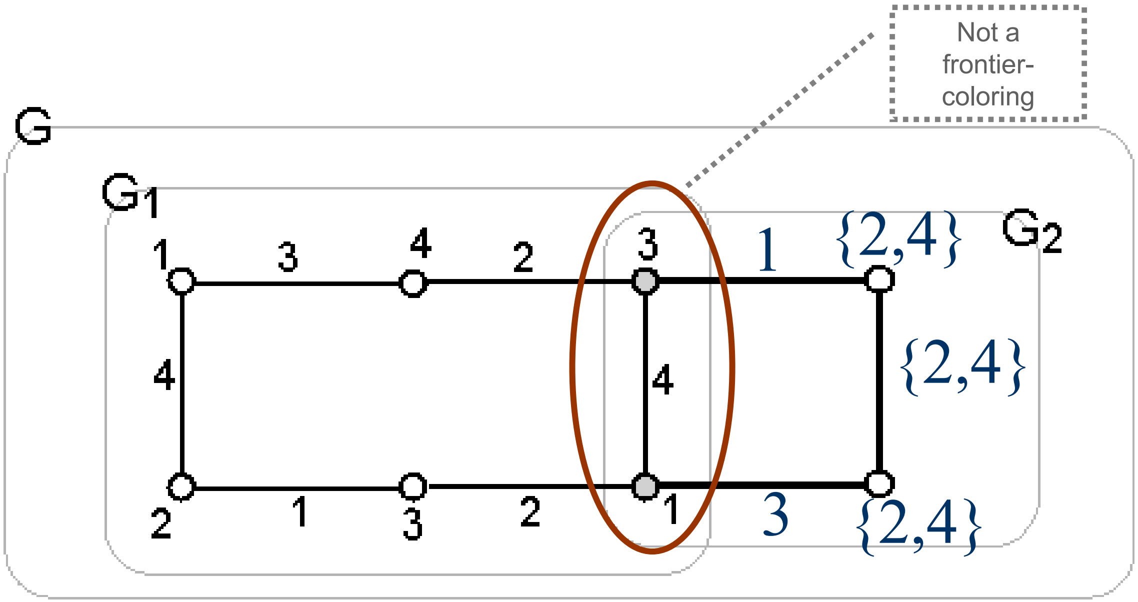


Decomposition for total-coloring: frontier-coloring

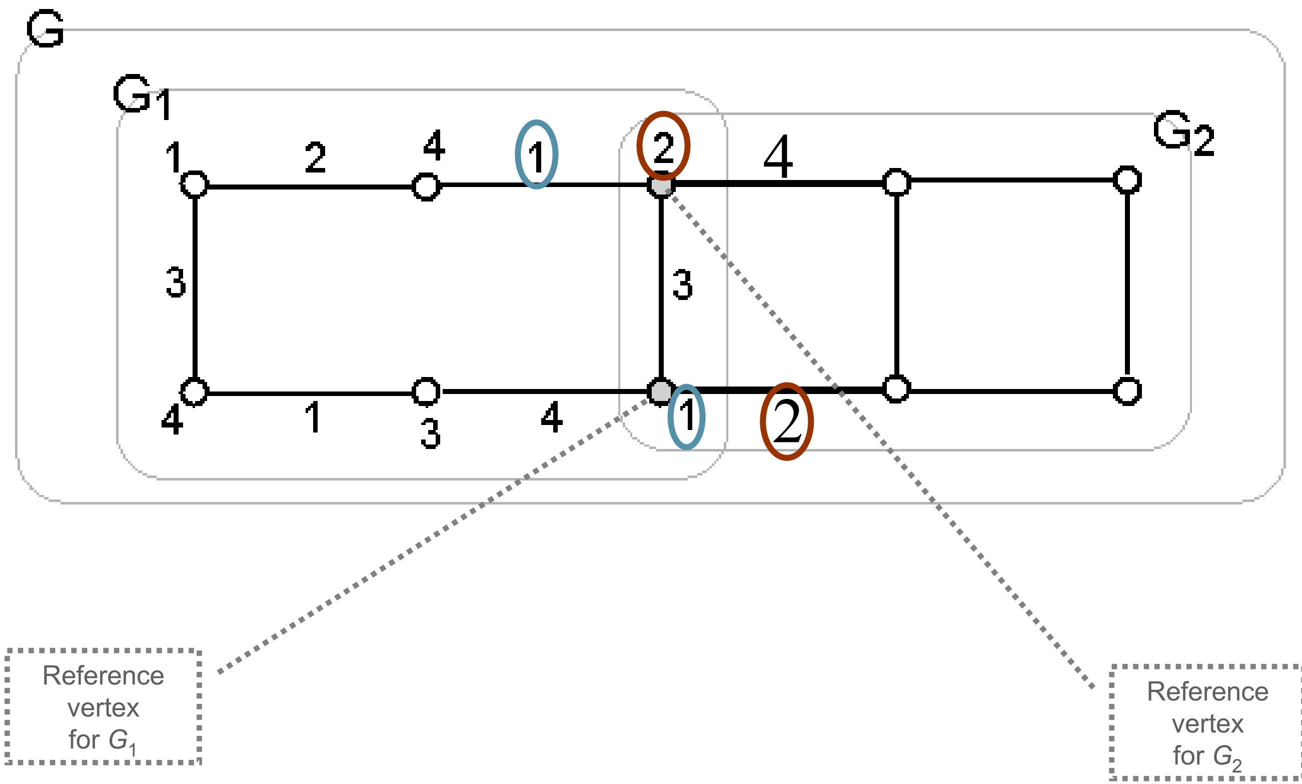
- A <u>frontier-coloring</u> is a coloring that satisfies the frontier condition for all frontier cadidates.



Why frontier-colorings?



Decomposition for total-coloring: frontier-coloring: “invertion” of reference vertices

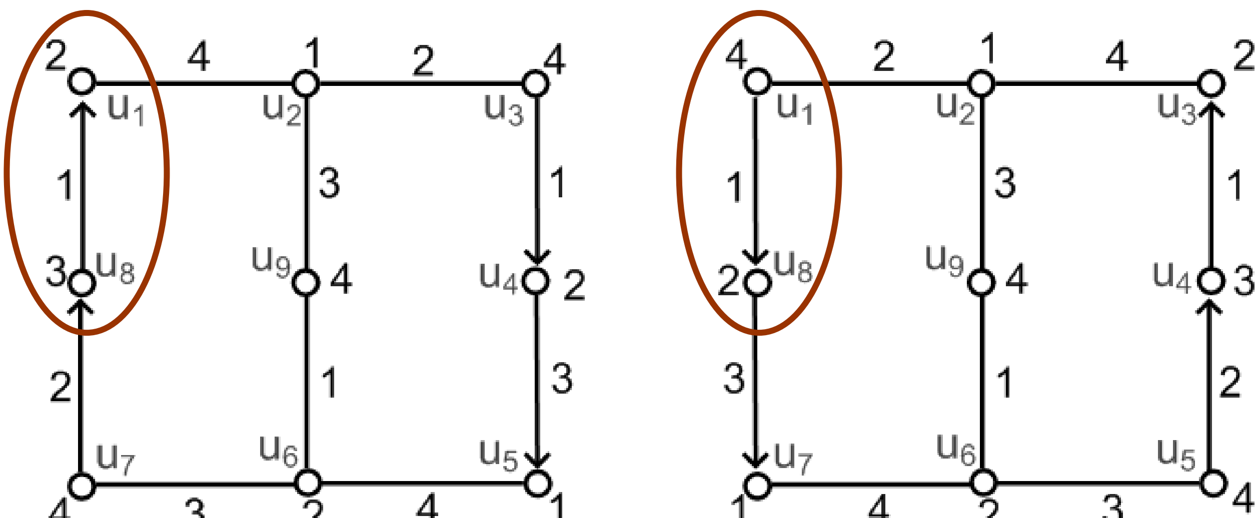


The decomposition result

- Consider a biconnected graph G of maximum degree 3.

  Suppose each K_2-cut-free component of G has two frontier-colorings π and π' such that, for each frontier-candidate {u,v}, u is reference vertex in π iff v is reference vertex in π'.

  In this case, G is 4-total-colorable.

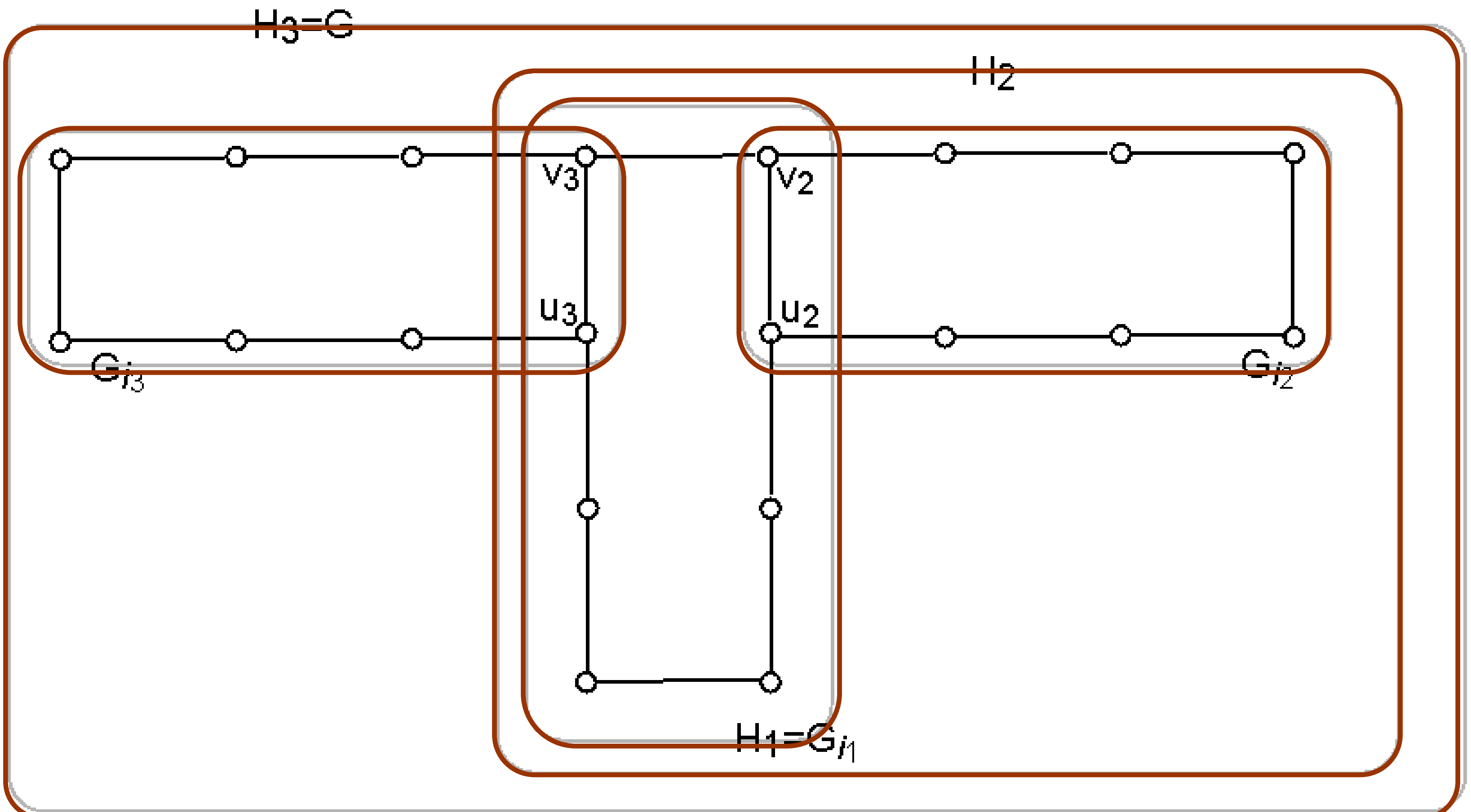


Decomposition for total-coloring: intersection graph of the K_2-cut-free components

- If G is a biconnected graph of maximum degree 3 and $\mathcal{G}$ is the collection of its K_2-cut-free components, then the intersection graph $\mathcal{I}(\mathcal{G})$ of $\mathcal{G}$ is acyclic.
- The above result allows us to 4-total-color G from 4-total-colorings of its K_2-cut-free components.

The decomposition

- Sketch of proof

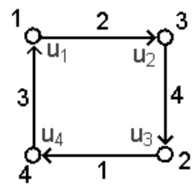


4-total-coloring partial-grids with bounded maximum induced cycle

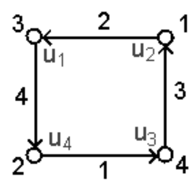
- A result: 8-chordal partial-grids with $\Delta=3$ are Type 1.
  - We just need to show frontier colorings for each P_2-cut-free partial-grid of maximum degree 3.
  - There is a finite number of these partial-grids.

The colorings...

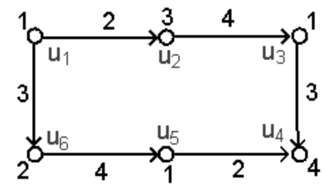
$\pi_{1,a}: S(G_1) \rightarrow \{1,2,3,4\}$



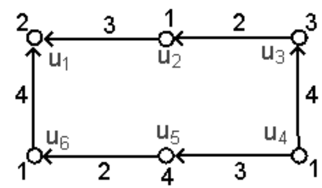
$\pi_{1,b}: S(G_1) \rightarrow \{1,2,3,4\}$



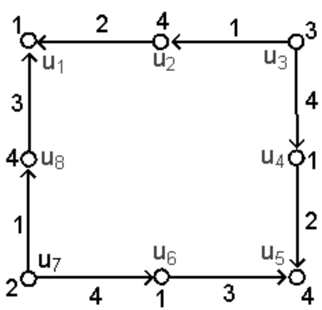
$\pi_{2,a}: S(G_2) \rightarrow \{1,2,3,4\}$



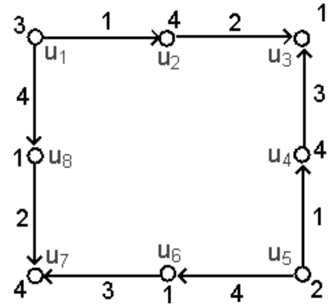
$\pi_{2,b}: S(G_2) \rightarrow \{1,2,3,4\}$



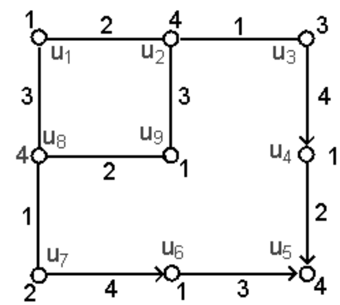
$\pi_{3,a}: S(G_3) \rightarrow \{1,2,3,4\}$



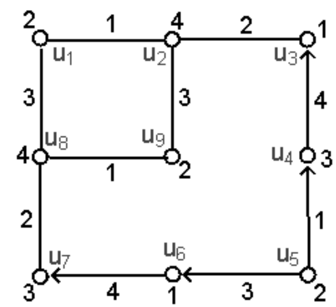
$\pi_{3,b}: S(G_3) \rightarrow \{1,2,3,4\}$



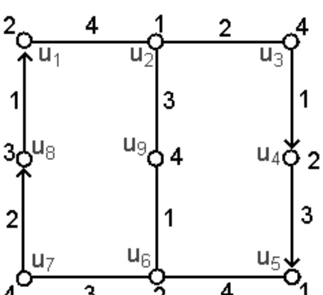
$\pi_{4,a}: S(G_4) \rightarrow \{1,2,3,4\}$



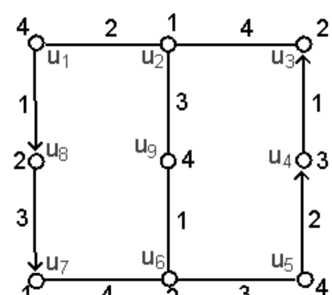
$\pi_{4,b}: S(G_4) \rightarrow \{1,2,3,4\}$



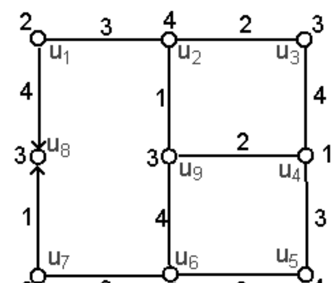
$\pi_{5,a}: S(G_5) \rightarrow \{1,2,3,4\}$



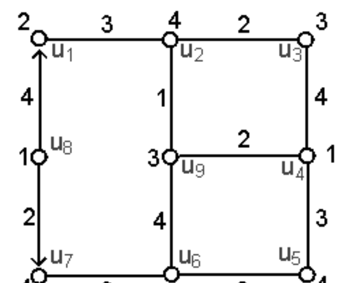
$\pi_{5,b}: S(G_5) \rightarrow \{1,2,3,4\}$



$\pi_{6,a}: S(G_6) \rightarrow \{1,2,3,4\}$



$\pi_{6,b}: S(G_6) \rightarrow \{1,2,3,4\}$



Another class: series-parallel graphs

- A graph is a SP-graph if it has no subgraph homeomorphic to K_4.
  - There are other possible recursive definitions

- Subclass of planar, the $\{K_5,K_{3,3}\}$-free graphs

- Superclass of outerplanar, the $\{K_4,K_{2,3}\}$-free graphs

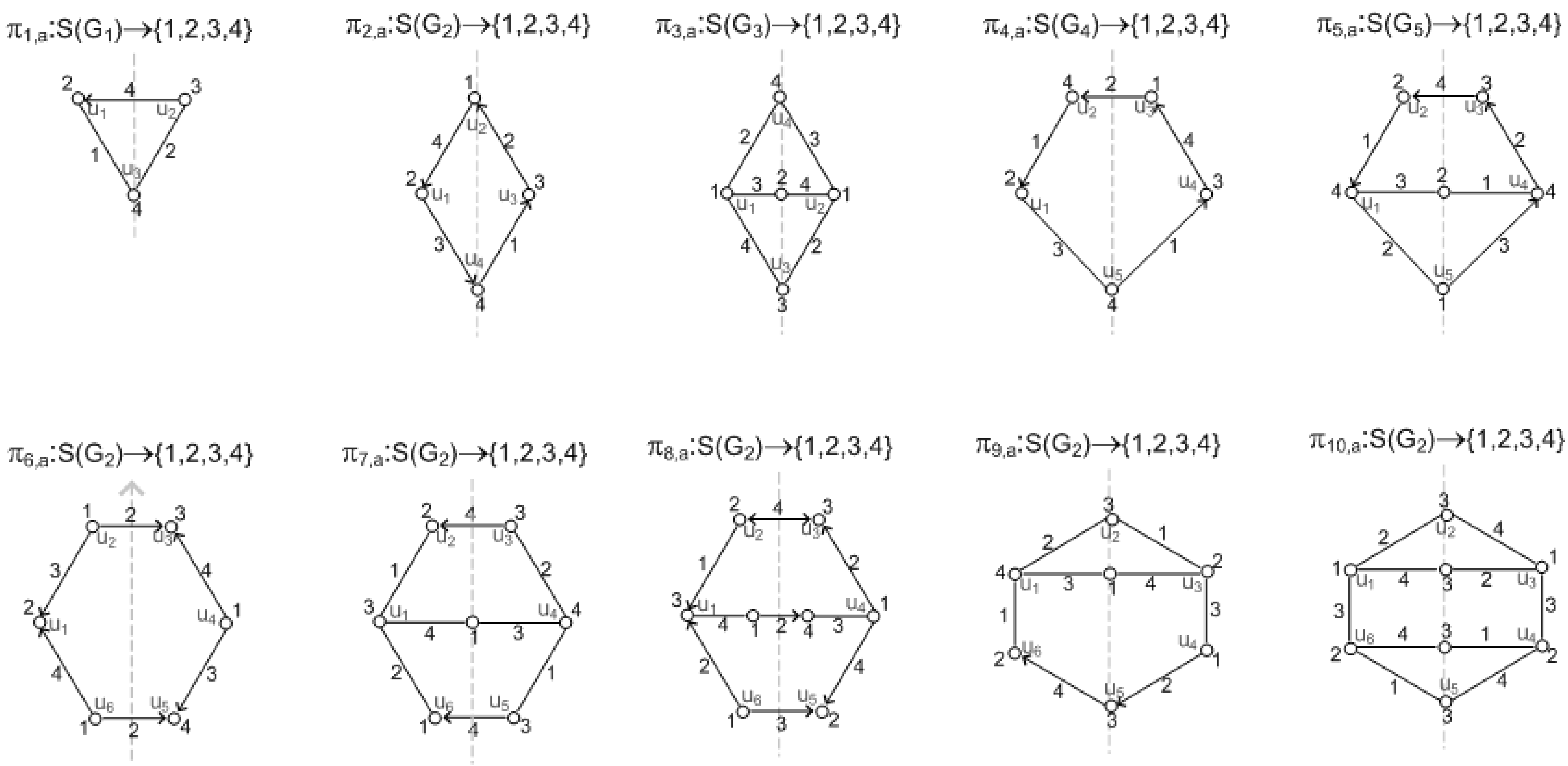
Total-coloring SP-graphs

- Every SP-graph of maximum degree $\Delta>3$ is Type 1
- The total chromatic number of graphs of maximum degree $\Delta = 1$ or 2 is easily determined
- The only open case is $\Delta = 3$
- We can apply our technique for subclasses with bounded maximum induced cycle.

4-total-coloring SP-graphs with bounded maximum induced cycle

- A result: 6-chordal partial-grids with $\Delta=3$ are Type 1.
  - We just need to show frontier colorings for each P_2-cut-free SP-graph of maximum degree 3.
  - There is a finite number of these SP-graphs.

The colorings...



Final considerations

- Our results
  - A decomposition for 4-total-coloring graphs of maximum degree 3.
  - Classification of a subset of partial-grids of maximum degree 3.
  - Similar result for SP-graphs
- Future goals
  - Writing a computer program for extending our results for partial-grids/SP-graphs with larger induced cycles.
  - Classification of all partial-grids.
  - Classification of all SP-graphs.

References

[1] M. Behzad. *Graphs and their chromatic numbers.* Doctoral Thesis Michigan State University (1965).

[2] A. Brandstädt, V. B. Le, T. Szymczak, F. Siegemund, H.N. de Ridder, S. Knorr, M. Rzehak, M. Mowitz, N. Ryabova, U. Nagel, D. Bayer. *Information System on Graph Class Inclusions v2.0* (http://wwwteo.informatik.uni-rostock.de/isgci/). Last visited 14 September 2007.

[3] G. Burosch and J.-M. Laborde. *Characterization of grid graphs.* Discrete Math. **87** (1991) 85-88.

[4] C. N. Campos, C. P. Mello. *The total chromatic number of some bipartite graphs.* In: 7th International Colloquium on Graph Theory, 2005, Marseille. Electron. Notes in Discrete Math. **22** (2005) 557-561. (Full paper to appear in Ars Combin.)

[5] D. G. Corneil, F. F. Dragan, E. Köhler. *On the power of BFS to determine a graph's diameter.* Networks **42** (2003) 209-222.

[6] C. J. H. McDiarmid, A. Sánches-Arroyo. *Total colouring regular bipartite graphs is NP-hard.* Discrete Math. **124** (1994) 155-162.

[7] S.-D. Wang and S.-C. Pang. *The determination of the total-chromatic number of series-parallel graphs.* Graphs and Combinatorics. **21** (2005) 531-540.

[8] V. G. Vizing. *On an estimate of the chromatic class of a p-graph.* Diskret. Analiz **3** (1964) 25-30. In Russian.

[9] Z. Zhongfu, Z. Jianxue, and W. Jianfang. *The total chromatic number of some graphs.* Scientia Sinica, Ser. A (1988) 1434-1441.

[10] B. Zmazek and J. Žerovnik. *Behzad-Vizing conjecture and Cartesian-product graphs.* Appl. Math. Lett. **15** (2002) 781-784.

Thank you