Global r-alliances and total domination

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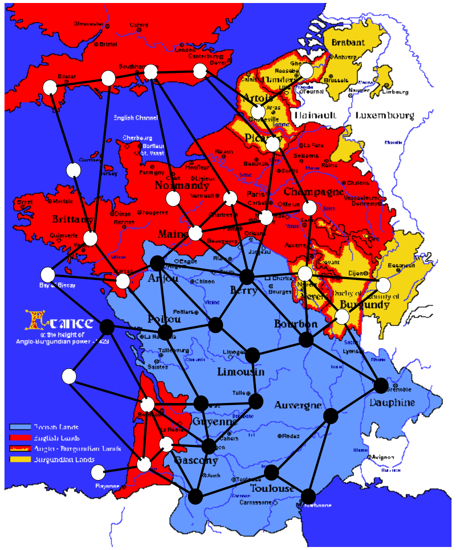
Motivation and Aim

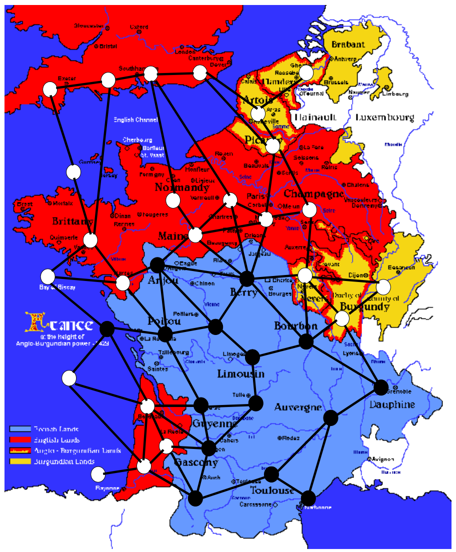
We consider nations that form alliances to defend themselves or to be able to attack other countries.



A graph-theoretic model, according to Hedetniemi *et al.*

- Nations are represented by vertices.
- Between each pair of nations, there is a bond (either modelling friendship or hostility).
- Nations can form different types of alliances.

Our Problem: An example



Regions that have many friends in the neighbourhood are less endangered than regions with few friends.
Conversely, regions that are surrounded by enemies are surele in danger.

Different types of alliances, according to Hedetniemi *et al.*

Defensive alliance

- Every member has at least as many bonds to other members (including itself) than to non-members.

- No member can be attacked successfully by non-members.

- Graph-theoretic formulation: $DA \subset V$ such that:
  for every $v \in DA$: $|N[v] \cap DA| \geq |N[v] \setminus DA|$.

$\boxed{\textbf{Different types of alliances}}$, according to Hedetniemi *et al.*

Offensive alliance

- Characterized by the vertices in their neighborhood outside of the alliance, written as $\partial OA := N[OA] \setminus OA$.

- Every such vertex has at least as many bonds to members in the alliance than to non-members (including itself).

- An offensive alliance can attack every neighbor successfully.

- graph-theoretic notation: $OA \subseteq V$, such that for every $v \in \partial OA$: $|N_G[v] \cap OA| \geq |N_G[v] \setminus OA|$ (boundary condition).

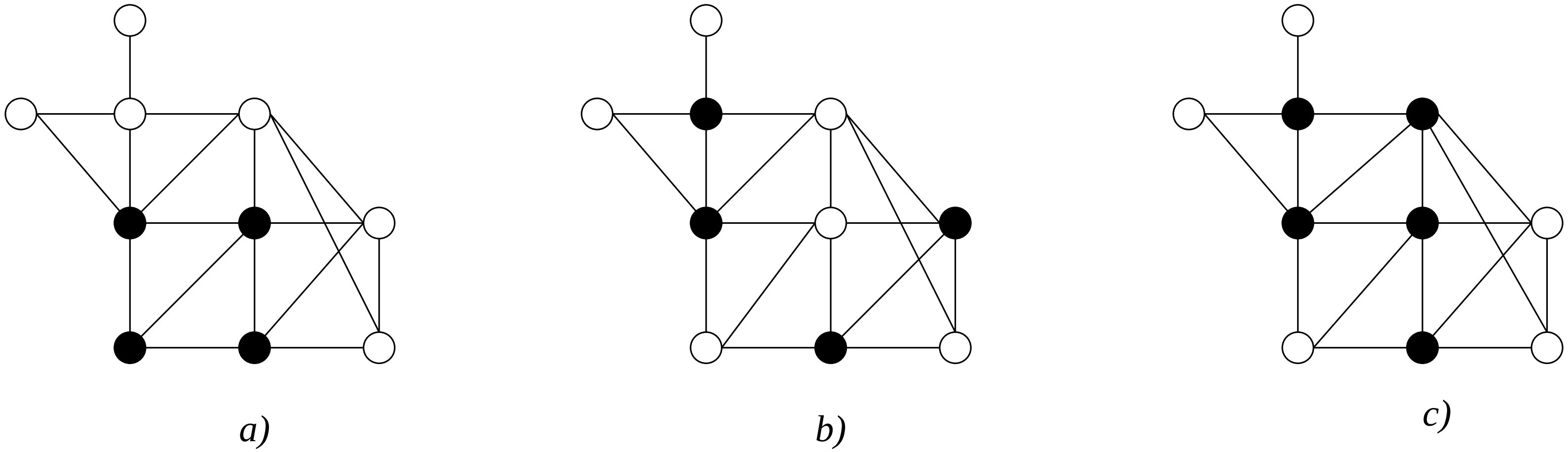
$\boxed{\textbf{Different types of alliances}}$, according to Hedetniemi *et al.*

- *Powerful* (or *dual*) alliances are both: defensive and offensive.

- Alliances are called *strong*, if the above inequalities are met strictly, leading to, e.g., strong defensive alliance.

- An Alliance is called *global*, if it is also a dominating set.

Examples



The black vertices form an alliance in each graph: $a)$ a defensive alliance $b)$ an offensive alliance $c)$ a powerful alliance.

$\boxed{r\text{-Alliances}}$ Notation: $\delta_A(v) = |\{u \in A \mid u \in N(v)\}|$.

J. A. Rodríguez and J. M. Sigarreta generalized the introduced concepts by introducing a slackness condition called *strength parameter* r.

—$S \subseteq V$, $S \neq \emptyset$, is called a *defensive r-alliance* if for every $v \in S$, $\delta_S(v) \geq \delta_{\overline{S}}(v) + r$. A defensive (-1)-alliance is a "*defensive alliance*".
—$S \subseteq V$ is called an *offensive r-alliance* if for every $v \in \partial S$, $\delta_S(v) \geq \delta_{\overline{S}}(v) + r$, where $-\Delta + 2 < r \leq \Delta$.
In particular, an offensive 1-alliance is an "*offensive alliance*".
—$S \subseteq V$ is a dual r-alliance if S is both a global defensive r-alliance and an $(r + 2)$-offensive alliance.
Graph-theoretic numbers (global!): γ_r^d, γ_r^o, γ_r^*

$\boxed{\textbf{Global } r\textbf{-Alliances}}$ will be in the focus of this presentation.

CTW history:

Note 1: ”Global offensive alliances in graphs“ CTW’06 (J.A.R. and J.M.S.)

Note 2: ”On the defensive k-alliance number of a graph“ CTW’07 (J.A.R. and J.M.S.)

Today’s focus:

(A) ”Global“, i.e., dominance aspects

(B) ”dual“, i.e., both defensive and offensive.

For the sake of simplicity of presentation, we also elaborate on ”defensive“ alliances.

Global defensive r-alliances

Cami *et al.* [1] showed NP-completeness for $r = -1$.

Theorem 1 *For all fixed r, the following problem is is NP-complete: Given a graph Γ and a bound ℓ; determine if $\gamma_r^d(\Gamma) \leq \ell$.*

Sketch: For $r \leq 3$, we can use the fact that any $(-r)$-GDA is a dominating set on cubic graphs, and that the dominating set problem is NP-hard on cubic graphs.
For $r = -2$, we can modify Cami *et al.*'s construction.
For $r \geq 0$, we can give a different reduction from DOMINATING SET.

Combinatorial Results

Theorem 2 *For any graph* Γ, $\frac{\sqrt{4n+r^2}+r}{2} \leq \gamma_r^d(\Gamma) \leq n - \left\lceil \frac{\delta_n - r}{2} \right\rceil$.

Theorem 3 *For any graph* Γ, $\gamma_r^d(\Gamma) \geq \left\lceil \frac{n}{\left\lfloor \frac{\delta_1 - r}{2} \right\rfloor + 1} \right\rceil$.

Corollary 4 *For any graph* Γ *of size* m *and maximum degrees* $\delta_1 \geq \delta_2$, $\gamma_r^d(L(\Gamma)) \geq \left\lceil \frac{m}{\left\lfloor \frac{\delta_1+\delta_2-2-r}{2} \right\rfloor + 1} \right\rceil$, *where* $L(\Gamma)$ *denotes the line graph of* Γ.

Combinatorial Results: Notes

For any graph Γ, $\dfrac{\sqrt{4n+r^2}+r}{2} \leq \gamma_r^d(\Gamma) \leq n - \left\lceil \dfrac{\delta_n - r}{2} \right\rceil$.

The upper bound is attained, for instance, for the complete graph $\Gamma = K_n$ for every $r \in \{1-n, \dots, n-1\}$.
The lower bound is attained, for instance, for the 3-cube graph $\Gamma = Q_3$, in the following cases: $2 \leq \gamma_{-3}^d(Q_3)$ and $4 \leq \gamma_1^d(Q_3) = \gamma_0^d(Q_3)$.

Global offensive r-alliances

Theorem 5 *For all fixed r, the following problem is NP-complete: Given a graph Γ and a bound ℓ; determine if $\gamma_r^o(\Gamma) \leq \ell$.*

Combinatorial properties have been presented at the previous CTW.
In addition, one can find interrelations with the concepts of *r-domination* (yielding the number γ_r) and the Laplacian spectral radius μ_*:

Theorem 6 *For any simple graph Γ of order n, minimum degree δ, and Laplacian spectral radius μ_*,* $\left\lceil \frac{n}{\mu_*} \left\lceil \frac{\delta+r}{2} \right\rceil \right\rceil \leq \gamma_r^o(\Gamma) \leq \left\lfloor \frac{\gamma_r(\Gamma)+n}{2} \right\rfloor$.

Global dual r-alliances; Some known examples

- $\gamma^*_{-1}(K_n) = \left\lceil \frac{n}{2} \right\rceil$.

- $\gamma^*_{-1}(P_n) = n - \left\lceil \frac{n}{3} \right\rceil$.

- $\gamma^*_{-1}(C_n) = n - \left\lfloor \frac{n}{3} \right\rfloor$.

- $p \leq s$, $\gamma^*_{-1}(K_{p,s}) = \min \left\{ \left\lceil \frac{p+1}{2} \right\rceil + \left\lceil \frac{s+1}{2} \right\rceil, p + \left\lfloor \frac{s}{2} \right\rfloor \right\}$.

- $\gamma^*_{-1}(W_n) = \left\lceil \frac{n+1}{2} \right\rceil$.

Global dual r-alliances

Theorem 7 *For all fixed r, the following problem is NP-complete: Given a graph Γ and a bound ℓ; determine if $\gamma_r^*(\Gamma) \leq \ell$.*

Theorem 8 *For any graph Γ of order n, size m and minimum degree δ,*

$$\left\lceil \frac{\sqrt{8m+4n(r+2)+(r+1)^2}+r+1}{4} \right\rceil \leq \gamma_r^*(\Gamma) \leq n - \left\lceil \frac{\delta - r}{2} \right\rceil .$$

Proof. If S is a global offensive $(r+2)$-alliance, then

$$\sum_{v\in \bar{S}} \delta_S(v) \geq \sum_{v\in \bar{S}} \delta_{\bar{S}}(v) + (n - |S|)(r+2). \tag{1}$$

Hence, as $\sum_{v\in S} \delta_{\bar{S}}(v) = \sum_{v\in \bar{S}} \delta_S(v)$,

$$\sum_{v\in \bar{S}} \delta_S(v) \geq \left(2m - \sum_{v\in S} \delta_S(v) - 2\sum_{v\in \bar{S}} \delta_S(v) \right) + (n - |S|)(r+2). \tag{2}$$

Thus,

$$3\sum_{v\in \bar{S}} \delta_S(v) + \sum_{v\in S} \delta_S(v) \geq 2m + (n - |S|)(r+2). \tag{3}$$

On the other hand, if S is a global defensive r-alliance in Γ,

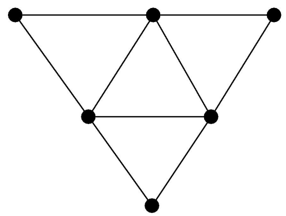
$$\sum_{v\in S}\delta_S(v)\geq\sum_{v\in S}\delta_{\overline{S}}(v)+r|S|. \tag{4}$$

Therefore, by (3) and (4) we have

$$4\sum_{v\in S}\delta_S(v)\geq 2m+n(r+2)+2s(r-1). \tag{5}$$

Thus, by $|S|(|S|-1)\geq\sum_{v\in S}\delta_S(v)$ and (5), the result follows. ▮

The lower bound is attained for $r=-1$ and $r=0$ in the case of the graph on the right hand side.



Total domination

We consider the following decidability problem total r-domination (r-TD) for each fixed integer $r \geq 1$:
Given $\Gamma = (V, E)$ and an integer parameter ℓ, is there a vertex set D with $|D| \leq \ell$ such that $\delta_D(v) \geq r$ for all $v \in V$?
The smallest ℓ such that Γ together with ℓ forms a YES-instance of r-TD is denoted $\gamma_{rt}(\Gamma)$.

Theorem 9 $\forall r \geq 1$*: r-TD is NP-complete.*

Reduction idea: Use the known result for $r = 1$, adding r new vertices to a 1-TD instance.

Total domination and global dual alliances

Theorem 10 *Every total k-dominating set is a global defensive (offensive) r-alliance, where $-\Delta < r \leq 2k - \Delta$. Moreover, every global dual r-alliance, $r \geq 1$, is a total r-dominating set.*

Proof.

1. If $S \subset V$ is a total k-dominating set in Γ and $r \leq 2k - \Delta$, then

$$\delta_S(v) \geq k \geq r + \Delta - k \geq r + \delta(v) - k \geq r + \delta_{\bar{S}}(v), \quad \forall v \in V.$$

   Therefore, S is both defensive r-alliance and offensive r-alliance in Γ.

2. If $S \subset V$ is a global defensive r-alliance, then $\delta_S(v) \geq \delta_{\bar{S}}(v) + r \geq r$, $\forall v \in S$. Moreover, if $S \subset V$ is a global offensive $(r+2)$-alliance, then $\delta_S(v) \geq \delta_{\bar{S}}(v) + r + 2 \geq r$, $\forall v \in \bar{S}$. Therefore, $\delta_S(v) \geq r$, $\forall v \in V$.

∎

Total domination and global dual alliances

Corollary 11 *Each total k-dominating set is a global dual r-alliance, where $-\Delta < r \leq 2(k-1) - \Delta$.*

Corollary 12

- *For $-\Delta < r \leq 2k - \Delta$, $\gamma_{kt}(\Gamma) \geq \gamma_r^d(\Gamma)$ and $\gamma_{kt}(\Gamma) \geq \gamma_r^o(\Gamma)$.*
- *For $-\Delta < r \leq 2(k-1) - \Delta$, $\gamma_{kt}(\Gamma) \geq \gamma_r^*(\Gamma)$.*
- *For $k \geq 1$, $\gamma_k^*(\Gamma) \geq \gamma_{kt}(\Gamma)$.*

By Corollary 12 we have that lower bounds for $\gamma_r^d(\Gamma)$, $\gamma_r^o(\Gamma)$ and $\gamma_r^*(\Gamma)$ lead to lower bounds for $\gamma_{kt}(\Gamma)$. Moreover, upper bounds for $\gamma_{kt}(\Gamma)$ lead to upper bounds for $\gamma_r^d(\Gamma)$, $\gamma_r^o(\Gamma)$ and $\gamma_r^*(\Gamma)$.

Concluding Remarks

—The complexity results (NP-completeness) shown for various types of global alliances hold in the non-global case, as well.

—One can show fixed parameter tractability for all mentioned alliance problems.

—However, the seemingly related problems of r-(total)-domination are W[2]-hard.

Thanks for your attention !

Theoretische

Informatik Trier

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