Percolation on sparse random graphs with given degree sequence

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Random failures on networks

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Question: What if the devices themselves failed?

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edge percolation process with retainment probability p.

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edge percolation process with retainment probability p.

- This is the classical Erdős-Rényi model of random graphs, a.k.a. $G_{n,p}$.

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> Such a component is a called a *giant component*.

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- If $p > \frac{1+\delta}{n}$, then with probability $\to 1$, as $n \to \infty$, the remaining graph has a (unique) giant component;
- if $p < \frac{1-\delta}{n}$, then all the components of the remaining graph have $O(\log n)$ vertices, with probability $1 - o(1)$.

Our setting

For any integer $n \geq 1$, we let $\mathbf{d}_n = (d_1, \ldots, d_n)$ be a vector of non-negative integers such that $\sum_{i=1}^{n} d_i$ is even.

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We consider the set of all *simple* graphs on the vertex-set $\{1, \ldots, n\}$ whose degree sequence is $\mathbf{d}_n$.

We let $G(\mathbf{d}_n)$ be a random graph uniformly chosen among all *simple* graphs whose degree sequence is $\mathbf{d}_n$.

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We let $\tilde{G}(\mathbf{d}_n, p)$ be the remaining graph.

The question we will try to answer is:

Is there a component with at least ϵn vertices in $\tilde{G}(\mathbf{d}_n, p)$, for some $\epsilon > 0$?

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$$p_c := \frac{L'(1)}{L''(1)}.$$

The main theorem

For each $n \geq 1$, we have a degree sequence $\mathbf{d}_n = (d_1, \ldots, d_n)$, such that $Max.Degree \leq n^{1/9}$ and $L''(1) > L'(1)$.

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When $d_1 = \cdots = d_n = d \geq 3$, then for both types of percolation

$$p_c = \frac{1}{d-1}.$$

(That was known for the case of edge percolation (A. Goerdt))

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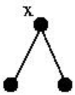

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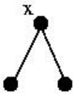

each one of its children generates independently a number of children distributed as X.

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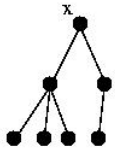

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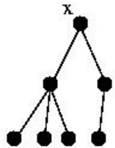

This continues for as long as there are "newborn" children.

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- if $E(X) < 1$, then with probability 1 the process dies out after a finite number of generations;
- if $E(X) > 1$, then the process goes on for an infinite number of generations, with positive probability.

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The expected number of children of v_1 is (almost equal to)

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The expected number of children of v_1 is (almost equal to)

$$\sum_{i\geq 1}(i-1)\frac{iD_i(n)}{D} \stackrel{n\to\infty}{\to} \sum_{i\geq 1} i(i-1)\frac{\lambda_i}{\sum_{j\geq 1} j\lambda_j} = \frac{L''(1)}{L'(1)},$$

where $D = \sum_{i\geq 1} iD_i(n)$.

Sketch of the proof

In fact, for all $d(v)$ neighbours of v this is true:

$$E\left(\sharp\text{children}\right) \to \sum_{i\geq 1} i(i-1)\frac{\lambda_i}{\sum_{j\geq 1} j\lambda_j} = \frac{L''(1)}{L'(1)}.$$

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a random tree where the expected number of children is approximately equal to

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So, locally $\tilde{G}(\mathbf{d}_n, p)$ will look like a random tree,
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Question:

Is $p_c = 0$ in this case?

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2. What is the rate of decrease of the giant component as $p \downarrow p_c$?

Thank you!