On the cardinality constrained matroid polytope

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Cardinality Constrained Combinatorial Optimization

- Combinatorial Optimization Problem (COP): $\Pi = (E, \mathcal{I}, w)$
  - ▷ E finite set
  - ▷ $\mathcal{I} \subseteq 2^E$ feasible solutions
  - ▷ w_e, $e \in E$, weighting

$$\max w(I) := \sum_{e \in I} w_e \quad \text{s.t. } I \in \mathcal{I}$$

- ... becomes *Cardinality Constrained Combinatorial Optimization Problem* (CCCOP): $\Pi_c = (E, \mathcal{I}, w, c)$
  - ▷ cardinality sequence $c = (c_1, \dots, c_m)$ with $0 \leq c_1 < \dots c_m \leq |E|$

$$\max w(I) \quad \text{s.t. } I \in \mathcal{I} \text{ and } |I| = c_p \text{ for some } p$$

Cardinality Constrained Combinatorial Optimization

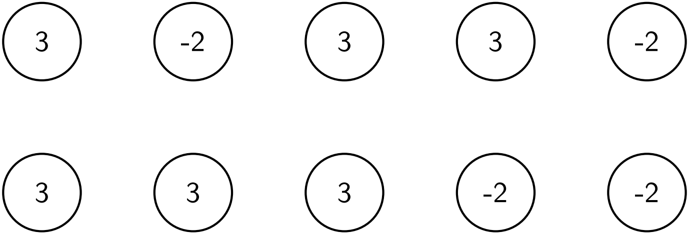
Given an IP-formulation for $\Pi = (E, \mathcal{I}, w)$, we obtain one for $\Pi_c = (E, \mathcal{I}, w, c)$ by adding

- ... the *cardinality bound* $c_1 \leq x(E) \leq c_m$
- ... Grötschel's *cardinality forcing inequalities*

$$(c_{p+1} - |F|)x(F) - (|F| - c_p)x(E \setminus F) \quad \leq \quad c_p(c_{p+1} - |F|)$$
$$\text{for all } \emptyset \neq F \subseteq E \text{ with } c_p < |F| < c_{p+1} \text{ for some } p \qquad \text{(CFI)}$$

Cardinality Forcing Inequalities

Example



counter 0

≤ 12

$$c = (3, 4, 9, 10), \ |F| = 6$$

Cardinality Forcing Inequalities

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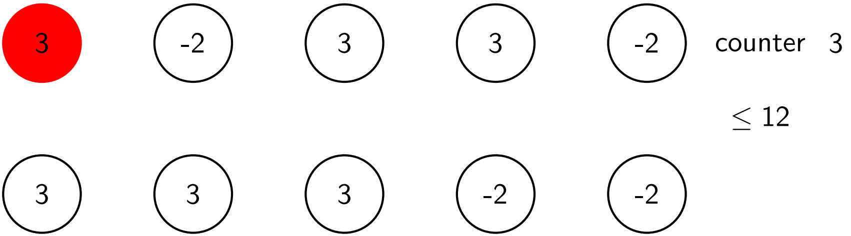

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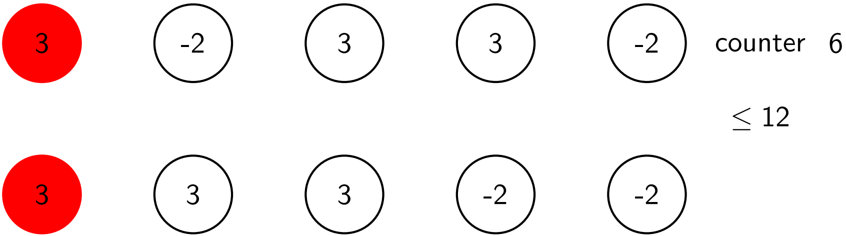

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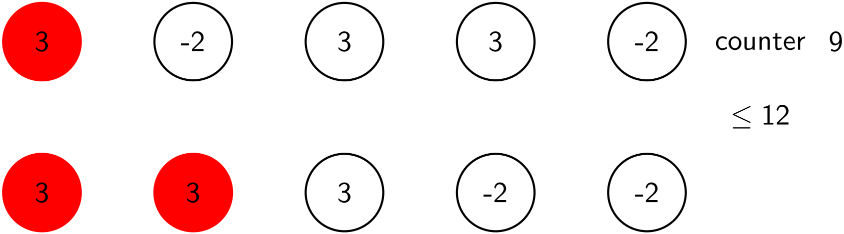

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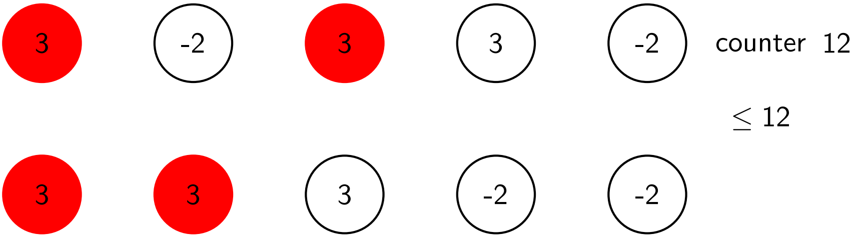

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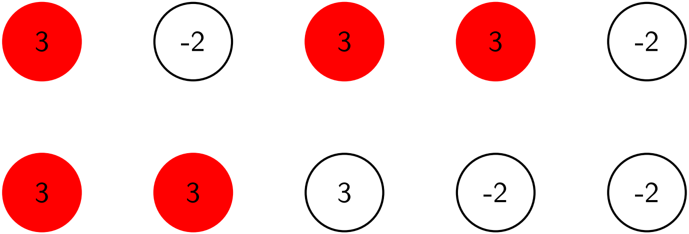

counter 15

≤ 12

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Cardinality Forcing Inequalities

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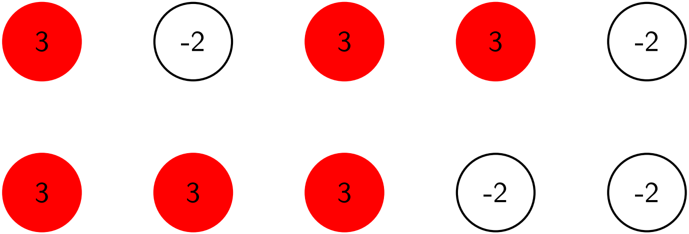

counter 18

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Cardinality Forcing Inequalities

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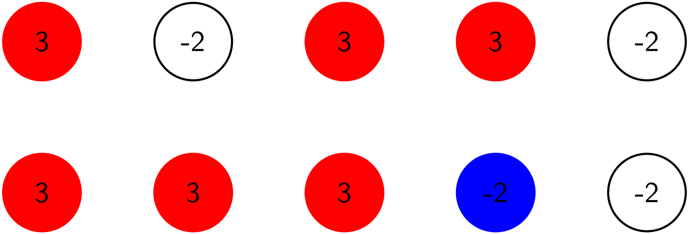

counter 16

≤ 12

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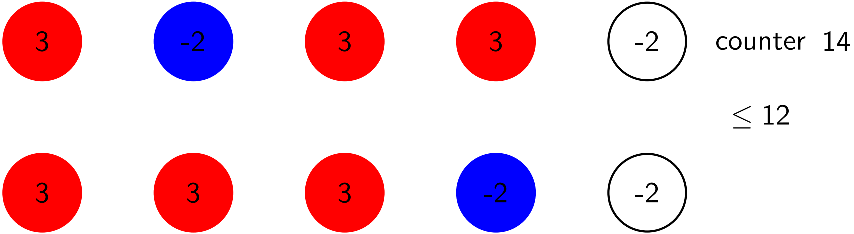
Cardinality Forcing Inequalities

Example



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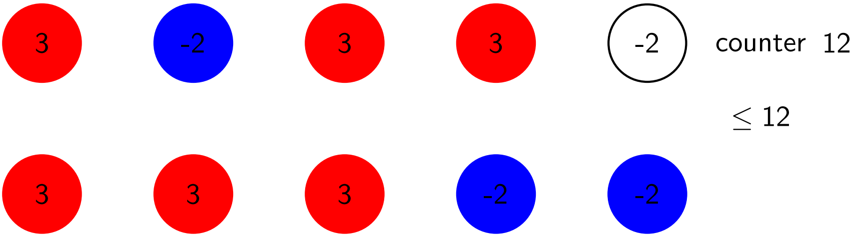

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Cardinality Constrained Combinatorial Optimization

If $\mathcal{I} = 2^E$, then ...

- ... Π_c is optimization problem over *cardinality homogeneous set systems* $\text{CHS}^c(E) := \{F \subseteq E : |F| = c_p \text{ for some } p\}$.
- ... associated polytope completely described by
  - ▷ nonnegativity constraints $x_e \geq 0$ for all $e \in E$,
  - ▷ cardinality bound $c_1 \leq x(E) \leq c_m$, and
  - ▷ cardinality forcing inequalities (CFI).

[M. Grötschel, *Cardinality homogeneous set systems, cycles in matroids*, ..., 2004]

Cardinality Constrained Combinatorial Optimization

+ CF-inequalities can be separated in polynomial time.
− In general: CF inequalities are quite weak inequalities.

Remedy: Study cardinality constrained matroid polytope.

Cardinality Constrained Matroid Polytope

From now on:

- $\Pi = (E, \mathcal{I}, w)$ maximum independent set problem over a matroid $\mathcal{I}$, that is,
  - (i) $\emptyset \in \mathcal{I}$,
  - (ii) $I \in \mathcal{I}, J \subseteq I \Rightarrow J \in \mathcal{I}$,
  - (iii) $I, J \in \mathcal{I}, |I| < |J| \Rightarrow e \in J \setminus I$ with $I \cup \{e\} \in \mathcal{I}$.
- Π_c card. constr. maximum independent set problem over a matroid $\mathcal{I}$

Definition

$$P^c_{\text{MAT}}(E) := \text{conv}\{\chi^I \in \mathbb{R}^E : I \in \mathcal{I} \cap \text{CHS}^c(E)\}$$

Cardinality Constrained Matroid Polytope

$r(F) = \text{rank of } F = \max\{|I| : I \in \mathcal{I}, I \subseteq F\}.$

$$x(F) \leq r(F) \quad \text{for all } \emptyset \neq F \subseteq E \qquad (1)$$

$$x_e \geq 0 \quad \text{for all } e \in E \qquad (2)$$

$$x(E) \geq c_1 \qquad (3)$$

$$x(E) \leq c_m \qquad (4)$$

$$(c_{p+1} - |F|)x(F) - (|F| - c_p)x(E \setminus F) \leq c_p(c_{p+1} - |F|)$$

$$\text{for all } \emptyset \neq F \subseteq E \text{ with } c_p < |F| < c_{p+1} \text{ for some } p \qquad \text{(CFI)}$$

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$$
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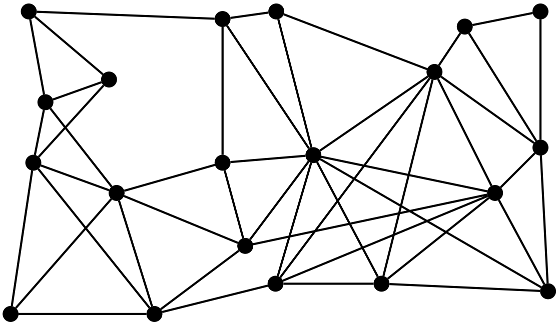
$$\overbrace{(c_{p+1} - r(F))x(F) - (r(F) - c_p)x(E \setminus F)}^{\text{CF}_F(x)} \leq c_p(c_{p+1} - r(F))$$

for all $\emptyset \neq F \subseteq E$ with $c_p < r(F) < c_{p+1}$ for some p (rCFI)

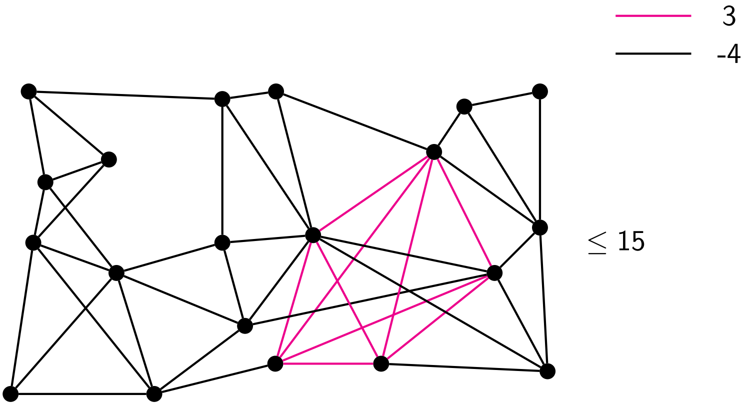
Example: Graphic Matroid

$c = (3, 5, 12, 14, 15, 18)$



Example: Graphic Matroid

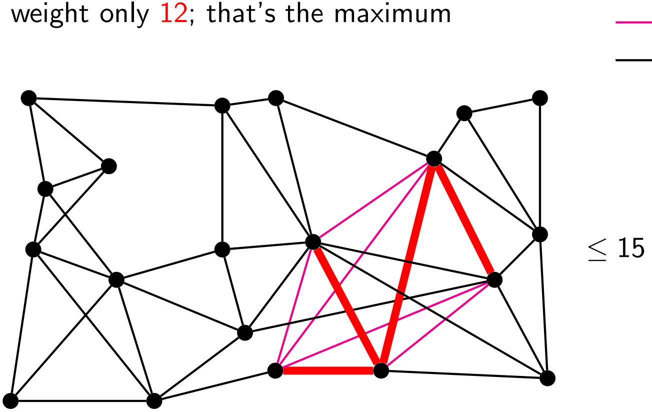
$c = (3, 5, 12, 14, 15, 18)$ $|F| = 9$



Example: Graphic Matroid

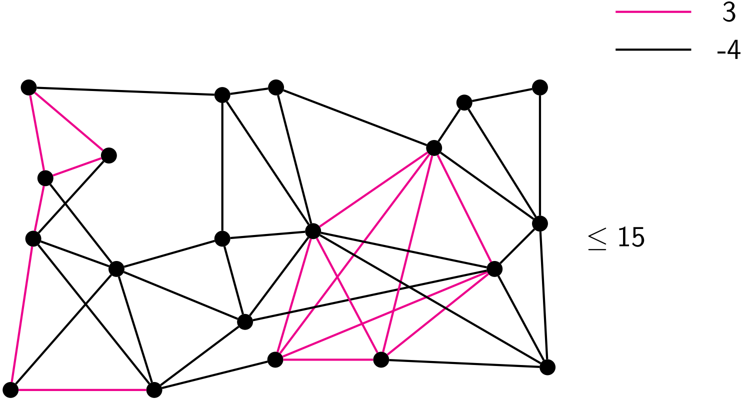
$c = (3, 5, 12, 14, 15, 18)$ $|F| = 9$. However ...

weight only 12; that's the maximum



Example: Graphic Matroid

$c = (3, 5, 12, 14, 15, 18)$ Better $r(F) = 9$ . . .



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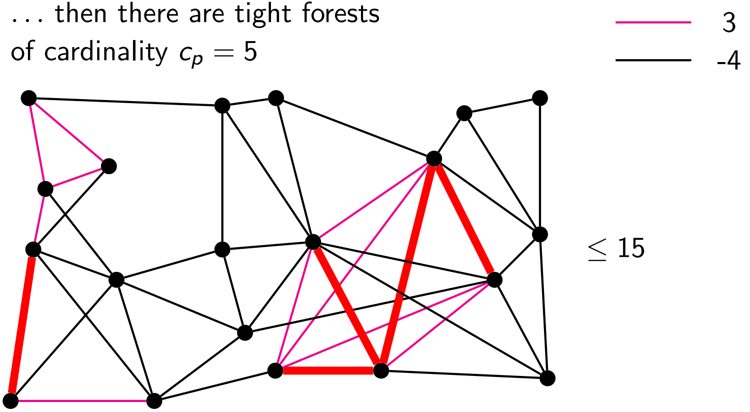

Example: Graphic Matroid

$c = (3, 5, 12, 14, 15, 18)$ Better $r(F) = 9$ ...

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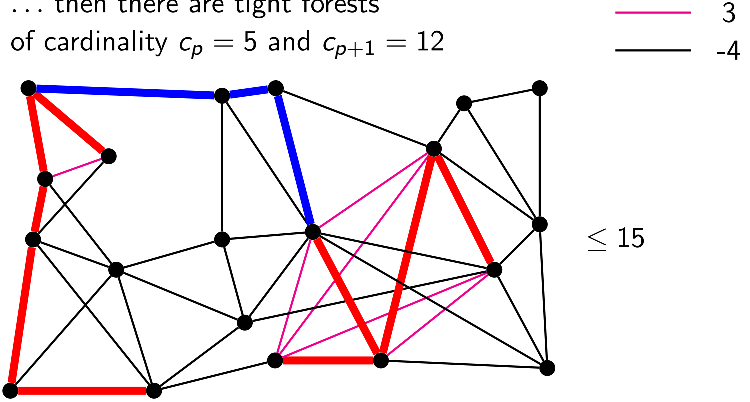

Cardinality Constrained Matroids

Results

- Π_c can be solved in polynomial time.
- System (1)-(4), (rCFI) provides a complete linear description of $P^c_{\text{MAT}}(E)$.
- In general: $\text{CF}_F(x) \leq c_p(c_{p+1} - r(F))$ defines a facet if and only if F is closed.
- Separation problem for CF-inequalities (rCFI) can be solved in poly-time.

Cardinality Constrained Matroids: Complexity

Theorem 1

Let $\mathcal{I}$ be a matroid. Then, Π_c can be solved in polynomial time.

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Proof.

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- doing this for all $k = c_p$, $p = 1, \ldots, m$ yields claim

Complete Linear Description

Theorem 2

$P^c_{\text{MAT}}(E)$ is determined by the inequalities

- rank inequalities (1)
- nonnegativity constraints (2)
- cardinality bounds (3) and (4)
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Sketch of proof.

All inequalities are valid $\Rightarrow$

$$P^c_{\text{MAT}}(E) \subseteq P := \{x \in \mathbb{R}^E : x \text{ satisfies } (1)\text{-}(4) \text{ and } (\text{rCFI})\}$$

Complete Linear Description

To show the converse:

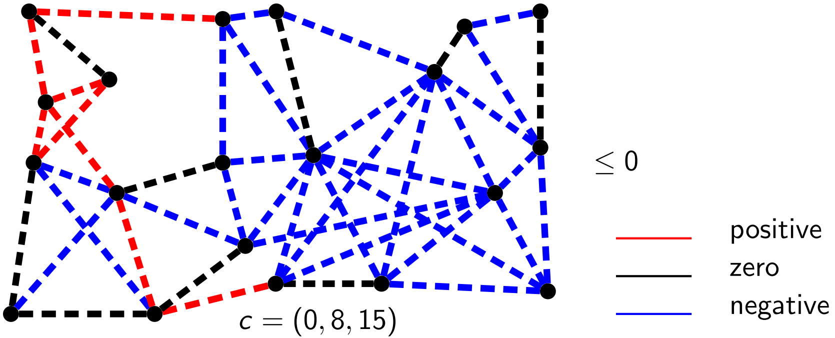
- Consider any valid inequality $bx \leq b_0$ for $P^c_{\text{MAT}}(E)$.
- Associate with $bx \leq b_0$ the following subsets of E:

$$
\begin{aligned}
P &:= \{e \in E : b_e > 0\}, \\
Z &:= \{e \in E : b_e = 0\}, \\
N &:= \{e \in E : b_e < 0\}.
\end{aligned}
$$

- Show by case by case enumeration on $\{P, Z, N\} \overset{\neq}{=} \emptyset$ and $b_0 \ \{<, =, >\}\ 0$ that the face F_b induced by $bx \leq b_0$ is contained in the face induced by some inequality among (1)-(4), (rCFI).
- By scaling argument, $b_0 \in \{-1, 0, 1\}$.

Illustration: Graphic Matroid, $b_0 = 0, c_1 = 0, P \neq \emptyset \neq N$

If $c_2 \leq r(P \cup Z)$ ...



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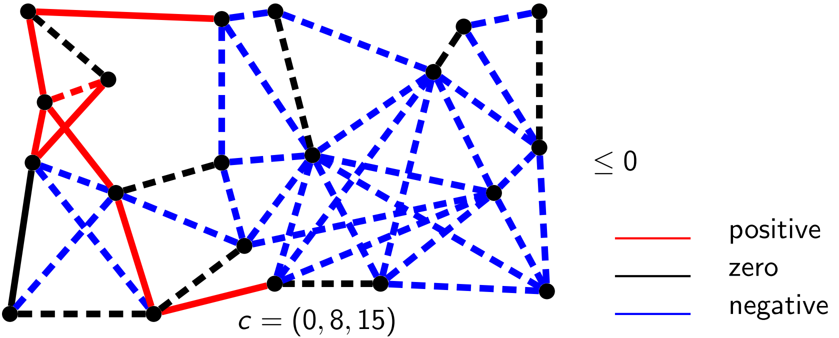
If $c_2 \leq r(P \cup Z)$ ...

... contradiction

$\Rightarrow c_2 > r(P \cup Z)$



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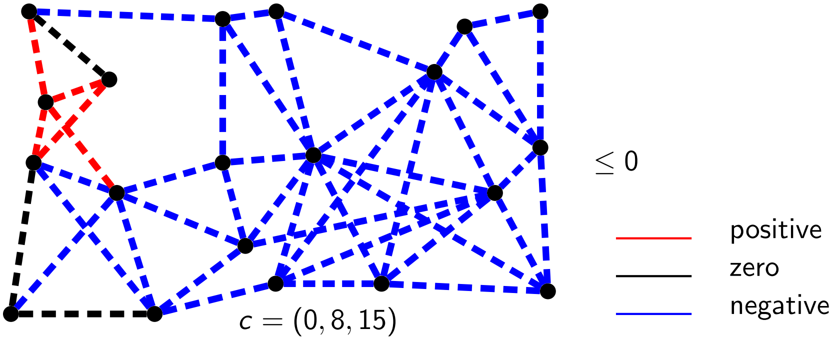

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If $b\chi^J = 0$, $|J| = c_p$, $p \geq 3$

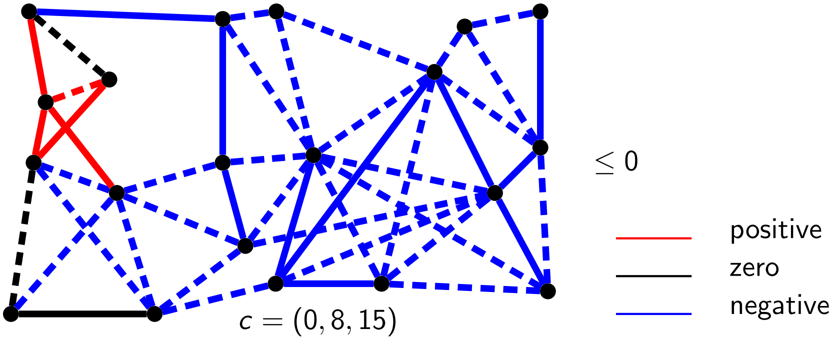
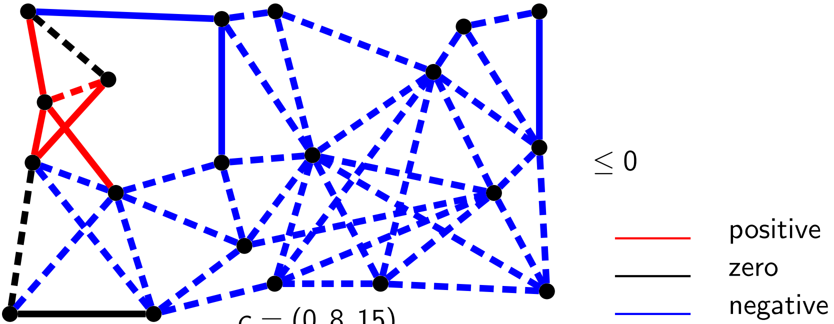


Illustration: Graphic Matroid, $b_0 = 0, c_1 = 0, P \neq \emptyset \neq N$

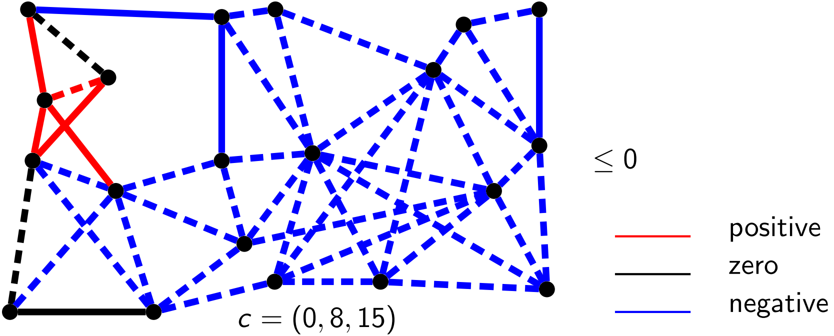
If $b\chi^J = 0$, $|J| = c_p$, $p \geq 3$,
$\Rightarrow \exists I \subset J, |I| = c_2 : b\chi^I > 0$



$c = (0, 8, 15)$

Illustration: Graphic Matroid, $b_0 = 0, c_1 = 0, P \neq \emptyset \neq N$

Thus, $\forall\ I$: $b\chi^I = 0 \Rightarrow |I| = 0$ or $|I| = c_2$
Assume $b\chi^I = 0$, $|I| = c_2$, but $|I \cap (P \cup Z)| < r(P \cup Z)$



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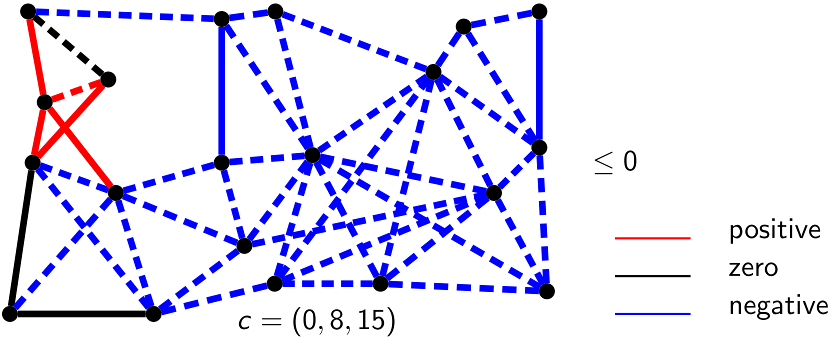

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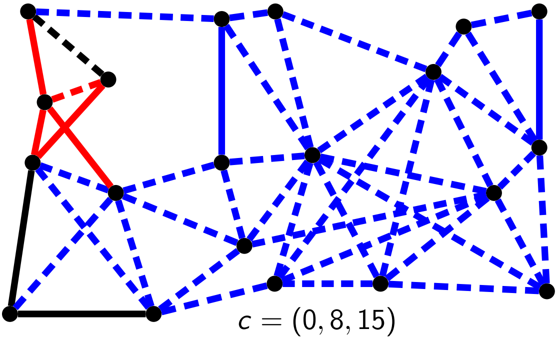
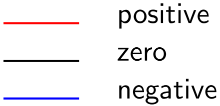
... contradiction

$\Rightarrow F_b$ contained in the face induced by $\text{CF}_{P \cup Z}(x) \leq 0$.



≤ 0



Facets

Definition

$F \subseteq E$ is said to be *closed* if $r(F \cup \{e\}) > r(F)$ for all $e \in E \setminus F$.

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Theorem 3

Let $F \subseteq E$ with $c_p < r(F) < c_{p+1}$.for some p. Moreover, let $c_p > 0$ and $c_{p+1} < r(E)$. Then, $\text{CF}_F(x) \leq c_p(c_{p+1} - r(F))$ defines a facet of $P^c_{\text{MAT}}(E)$ if and only if F is closed.

Facets

Illustration: graphic matroid

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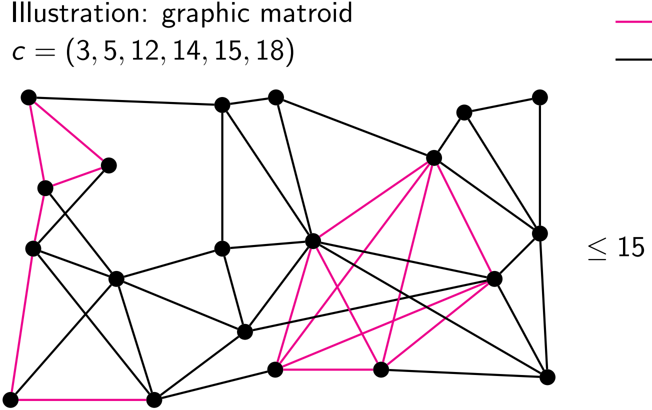

colored edge set not closed

Facets

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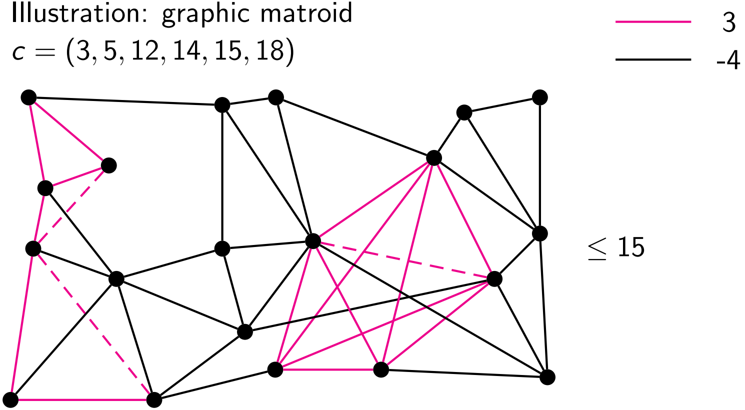

colored edge set closed

Separation Problem for $P^c_{\text{MAT}}(E)$

Given $x^* \in \mathbb{R}^E$, find valid inequality for $P^c_{\text{MAT}}(E)$ which is violated by x^*, or assert that $x^* \in P^c_{\text{MAT}}(E)$.

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- $\max\ w^T x,\ x \in P^c_{\text{MAT}}(E)$ can be solved in poly-time

$\Rightarrow$ Separation problem for $P^c_{\text{MAT}}(E)$ can be solved in poly-time (polynomial time equivalence of separation and optimization, see Grötschel, Lovász, Schrijver, 1988)

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What can we do in practice?

Separation Problem for Rank Inequalities

Given $x^* \in \mathbb{R}^E$, $x^* \geq 0$, find rank inequality $x(F) \leq r(F)$ violated by x^*.

- Can be solved in poly-time with an algorithm of Cunningham that maximizes $x(F) - r(F),\ F \subseteq E$.

Separation Problem for Cardinality Forcing Inequalities

- Trace back the separation problem for CF-inequalities to that for the rank inequalities!

Theorem 4

For any $x^* \in \mathbb{R}^E_+$ satisfying all rank inequalities (1), the separation problem for x^* and the cardinality forcing inequalities (rCFI) can be solved in polynomial time.

Separation Problem for Cardinality Forcing Inequalities

Proof idea. Compute $x^*(E)$.

- If $x^*(E) = c_p$ for some p, then
  $x^* \in P^{(c_p)}_{\text{MAT}}(E) \;\Rightarrow\; x^* \in P^{c}_{\text{MAT}}(E)$.
- If $c_p < x^*(E) < c_{p+1}$ for some p, then set $k := c_p$, $\ell := c_{p+1}$.
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- Sometimes good: Trace back CCCOP's to COP's.