Upward Embedding on T_h

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Outline

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- Horizontal Torus (T_h)

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- Upward embedding on torus (T_h and T_v tori)

Upward Embedding on T_h
- Equivalence Relation
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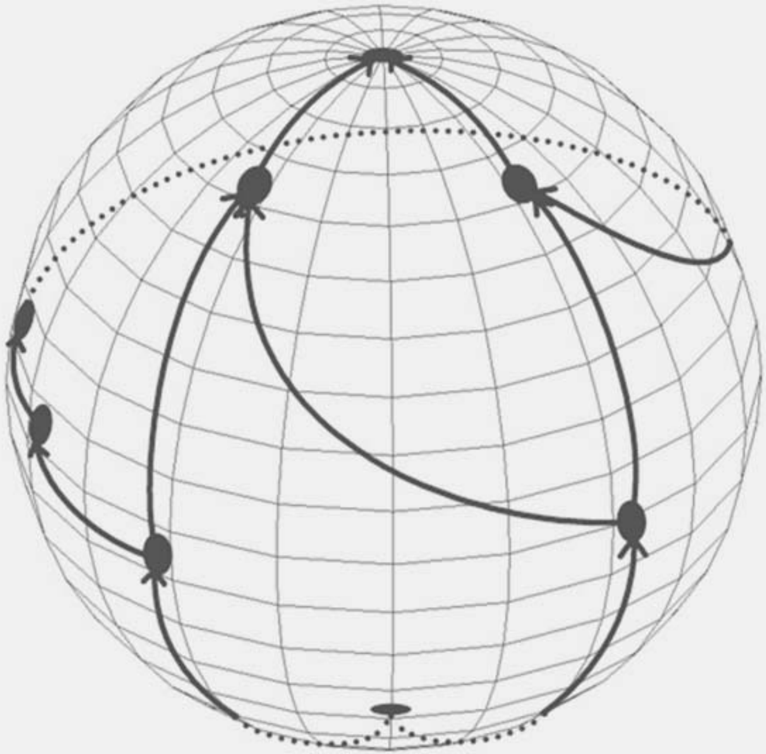
  - Testing
- All digraphs
  - Characterization
  - Testing

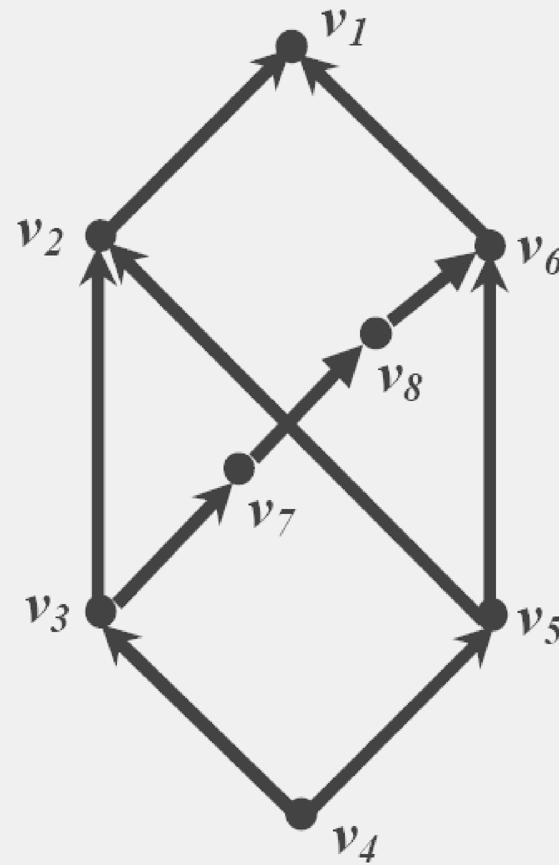
Open problems

Definition

An upward embedding of a digraph (directed graph) on the plane or a surface is an embedding of its underlying graph so that all directed edges are monotonic and point to a fixed direction. Such embedding in some literature is called upward drawing without crossing of edges

An example

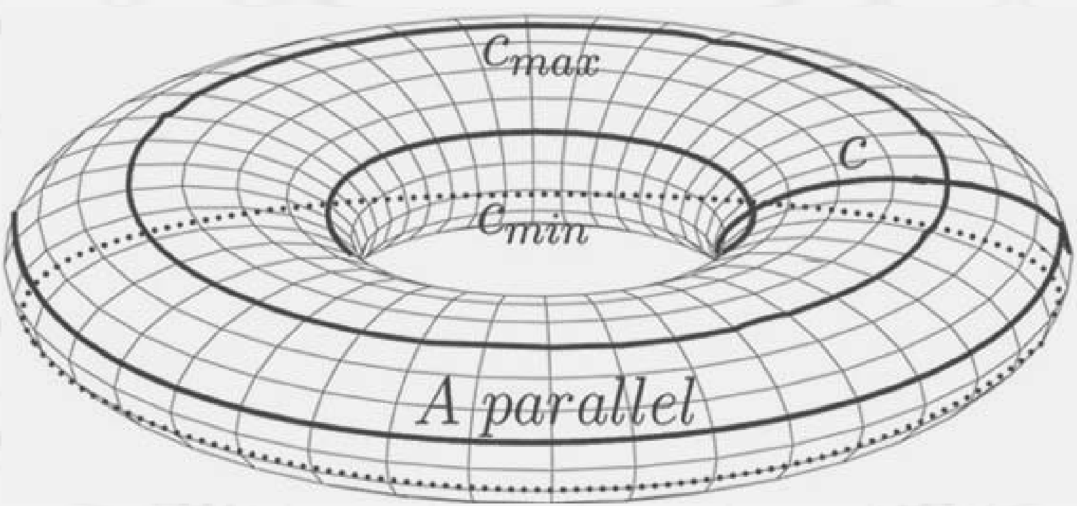




upward embedding of a digraph on sphere

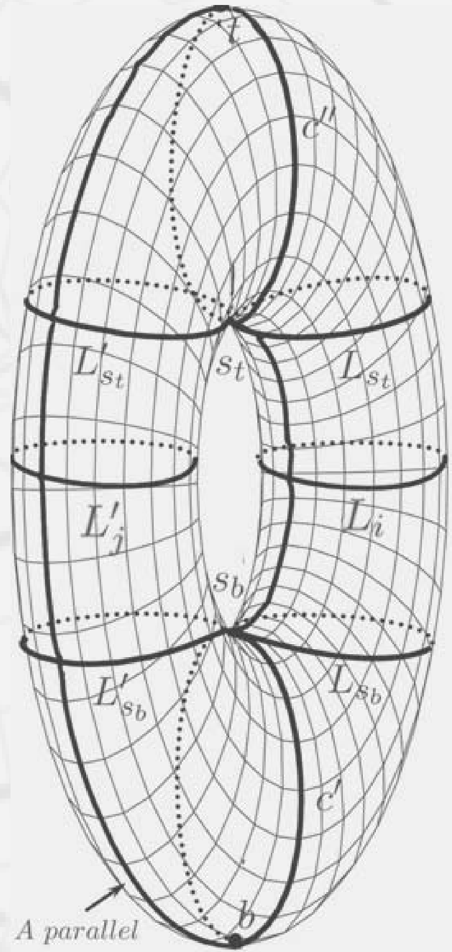
Definition

We define the horizontal torus $\boldsymbol{T_h}$ as the surface obtained by revolution of th curve $c : (y-2)^2 + (z-1)^2 = 1$ round the line $L : y = 0$ as its axis of revolution i the yz-plane. In this case we refer as inner layer to that part of T_h resulting fro the revolving of the part of c in which $y \le 2$. The other part of T_h resulting fro the revolving of that part of c in which $y \ge 2$ is called outer layer.



Definition

We also define a vertical torus $\boldsymbol{T_v}$ as the surface of revolution that is
resulted of revolving the curve
$C' : (x-1)^2 + (z-1)^2 = 1$
round the line $L_0 : z = 3$ in the xz-plane. In this case $b = (1, 0, 0)$ is the single minimum point of $\boldsymbol{T_v}$, $s_b = (1, 0, 2)$ and $s_t = (1, 0, 4)$ are its saddle pints, and $t = (1, 0, 6)$ is its single maximum.



Previous works for Plane

Polynomial Algorithms

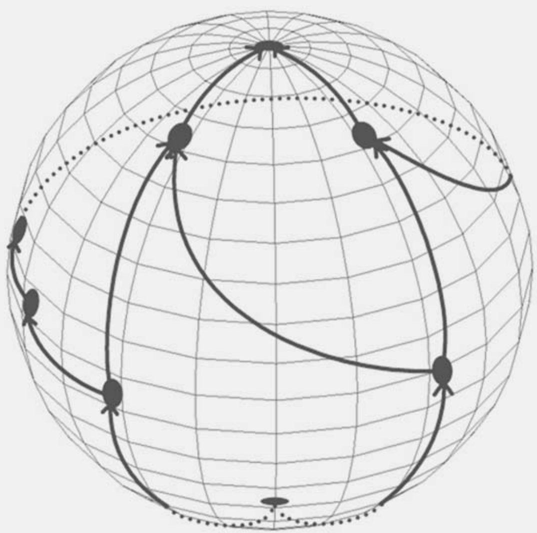
- *Triconnected digraphs*
  - P. Bertolazzi, G. Di Battista, G. Liotta, C. Mannino,. (1994)
- *single-source digraphs*
  - M.D. Hutton, A. Lubiw (1992)
  - P. Bertolazzi, G. Di Battista, C. Mannino, R. Tammasia, (1998)
- Outerplaner digraphs
  - A. Papakostas (1994)

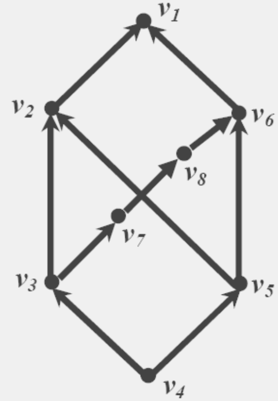
Upward embedcding testing on plane is an NP-complete problem.

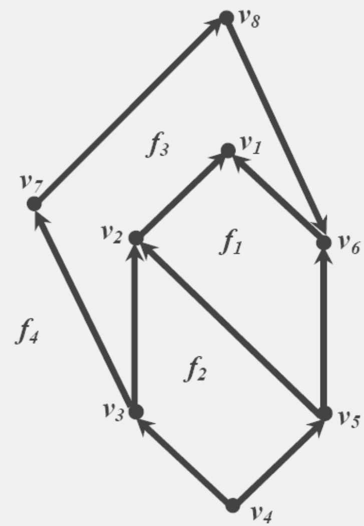
- A. Garg, R. Tammasia (1994)
- S. M. Hashemi, A. Kisielewicz, I. Rival (1998)

Previous works for Sphere

it has been proved that for upward embedding, plane and sphere are not equivalent which is in contrast with the fact that they are equivalent for undirected graphs.







Previous works for Sphere cont.

Polynomial Algorithms

- Embedded single source digraph
  - A. Dolati, S. M. Hashemi (2008)

Chahracterization of all digraphs that admit upwarde embedding

- S.M. Hashemi (2001)

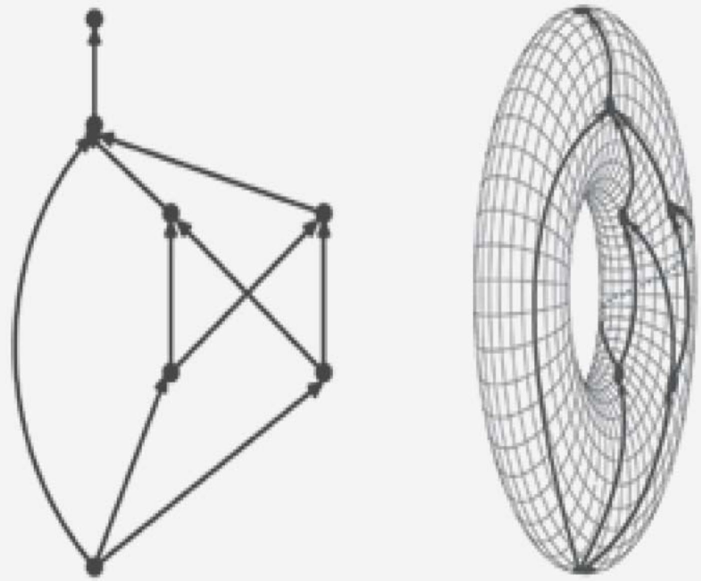
Upward embedcding testing on sphere is an NP-complete problem.

- S. M. Hashemi, A. Kisielewicz, I. Rival (1998)

Previous works for Torus

In spite of the equivalence of the tori for undirected graphs, they are not equivalen for upward embedding.

Consider the following digraph and an its upward embedding on T_v. This digrap does not have an upward embedding on T_h.



Previous works for Torus

- Polynomial Algorithms on T_h
  - *Single source and single sink digraphs*
    - A. Dolati, S. M. Hashemi, M. Khosravani (2008)

- **Theorem** (A. Dolati, S. M. Hashemi, M. Khosravani, 2008)

If a digraph D has an upward embedding on the horizontal torus T_h then it has an upward embedding on the vertical torus T_v.

Upward Embedding on T_h

Definition: Given a digraph $D = (V,A)$. We say two arcs a, a_0 of $A(D)$ are related by relation R denoted by aRa_0 if they belong to a directed path or there is a sequence $P_1, P_2, \ldots, P_k (k>1)$ of directed paths with the following properties:

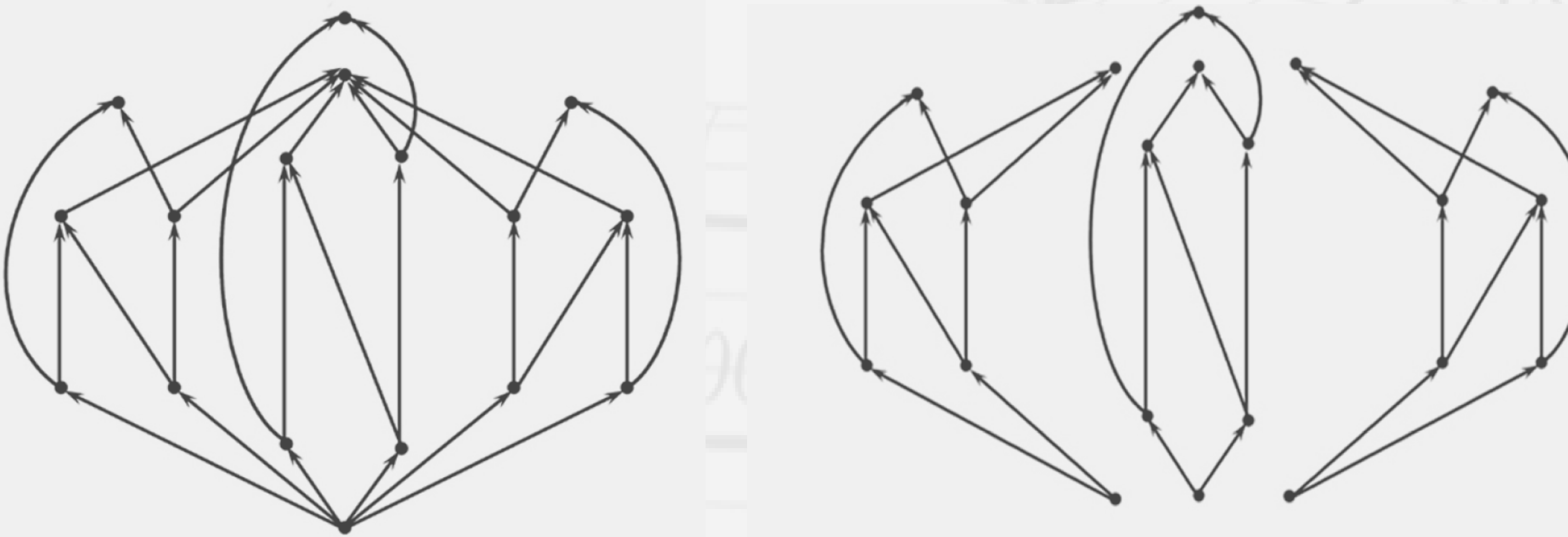
(i) a is an arc of P_1 and a_0 is an arc of P_k.

(ii) Every P_i, $i = 1, \ldots, k-1$ has at least one common vertex with P_{i+1} which is an internal vertex

R is an equivalence relation.

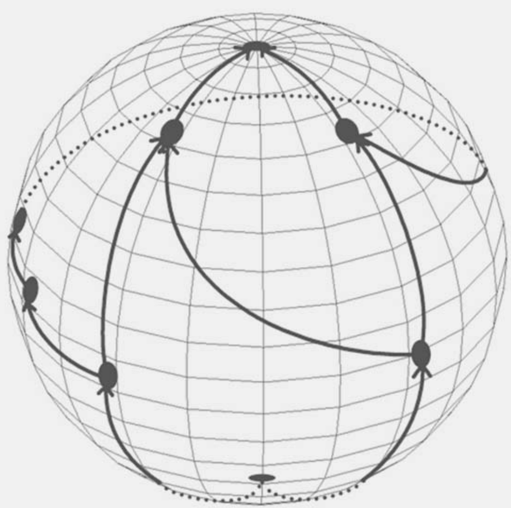
An example

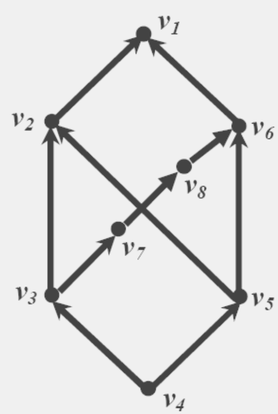
Theorem: Given a digraph D. In every upward embedding of D on T_h, all arcs that belong to the same class must be drawn on the same layer.

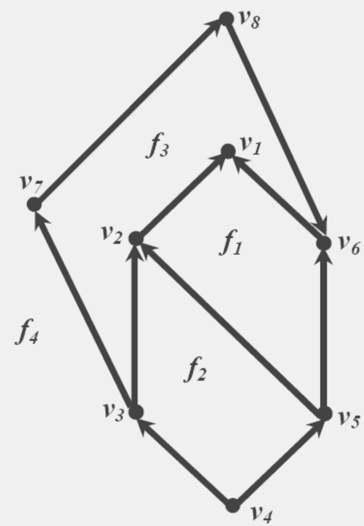


SNP-graph

- A digraph that has upward embedding on sphere but has no upward embedding on the plane.







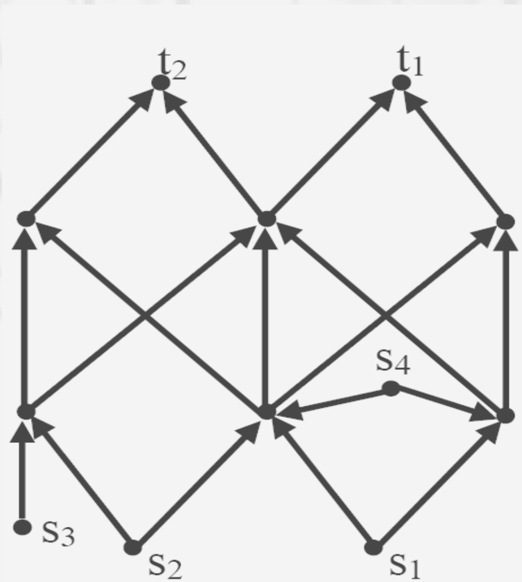
Theorem (**A. Dolati, S. M. Hashemi, M. Khosravani (2008)**)

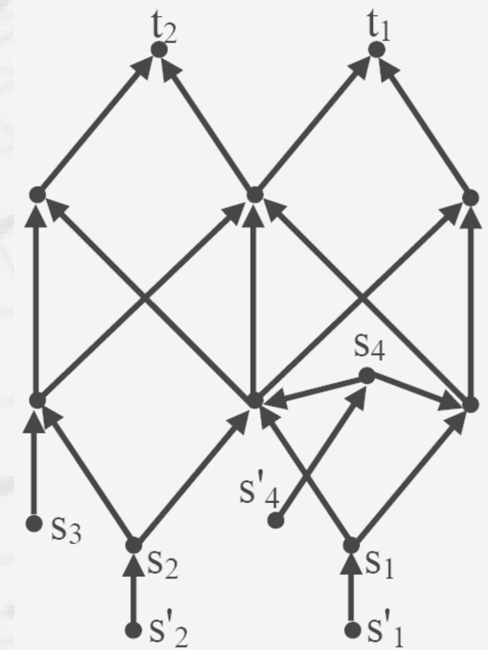
Suppose that D is a single source and single sink acyclic digraph and let $C_1, C_2, \ldots, C_k$ be the equivalence classes of its arcs with respect to the relation R. Also suppose that the digraphs $D_1, D_2, \ldots, D_k$ are the induced subdigraphs on $C_1, C_2, \ldots, C_k$, respectively. The digraph D has an upward embedding on T_h if and only if the underlying graphs of D_is are planar and either $k \leq 2$ or there is only one of D_is is an *SNP-digraph*.

Theorem 1. A digraph has an upward embedding on T_h if and only if by adding new arcs, if necessary, it can be extended to an acyclic single source and single sink digraph whose subdigraphs induced on the equivalence classes of its arcs with respect to R are planar and either at most one of them is an *SNP-digraph* or the number of equivalence classes is at most 2.

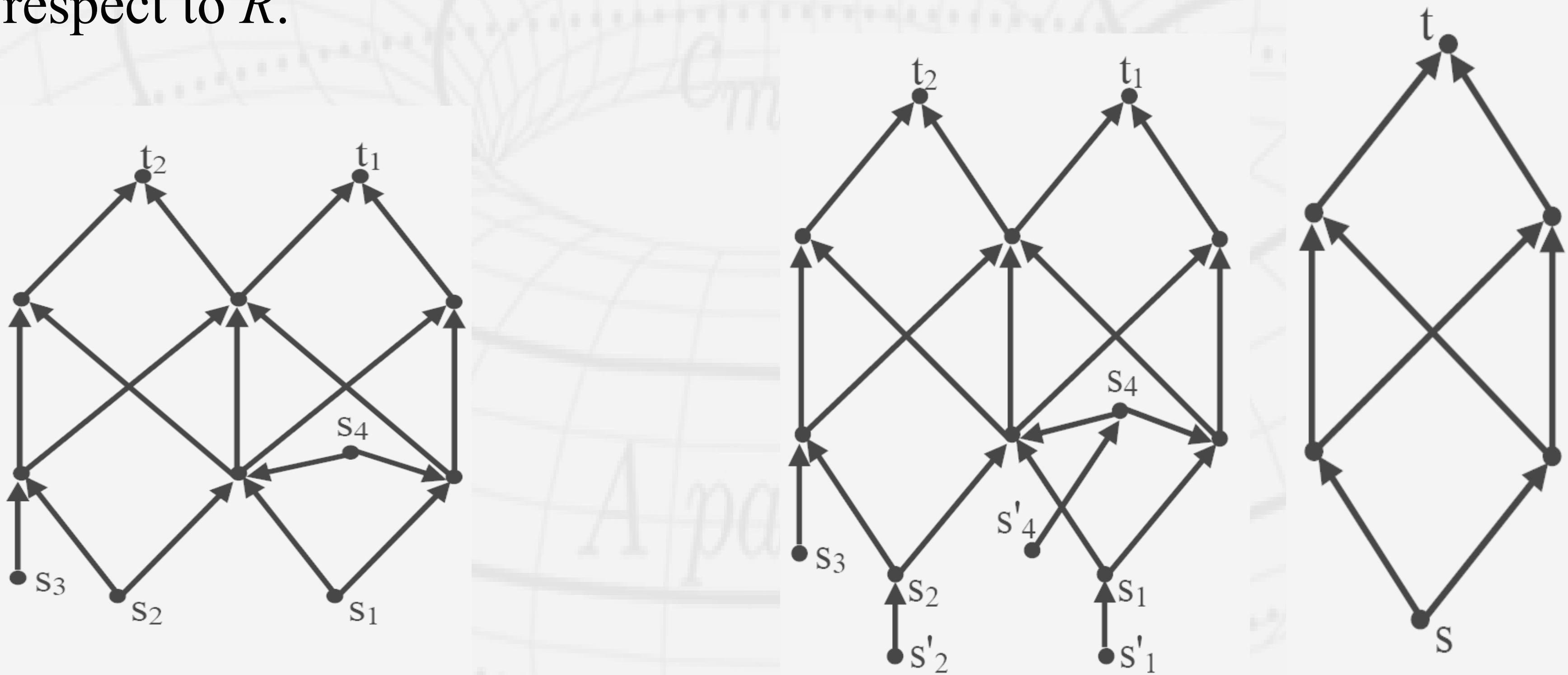
Definition of source-in-graph

suppose that $D = (V,A)$ is a digraph. Let $S=\{s_{i1}, \ldots, s_{im}\}$ be the set of its sources whose outgoing arcs are more than one. To build *source-in-graph* $SI(D)$ from D, we add the set of vertices $\{s'_{i1}, \ldots, s'_{im}\}$ and the set of arcs $\{(s'_{ij}, s_{ij}) | j = 1, \ldots, m\}$ to it.





The set of arcs of *SI(D)* has an unique equivalence class with respect to *R*.



Theorem. Suppose that D is a digraph and D' is a single source and single sink *SNP-digraph* whose source and sink are s and t, respectively. Let S and T be the set of sources and the set of sinks of $SI(D)$ respectively. D has an upward embedding on the sphere if and only if there exist s' of S and t' of T so that the resulting digraph from identifying sources s and s' and identifying sinks t and t' of D' and $SI(D)$ has an upward embedding on T_h.

The sphericity testing of a digraph can be done by upward embedding testing on T_h of $|S||T|$ digraphs where S is the set of the sources whose outgoing arcs are more than one and T is the set of the sinks.

Corollary. It is not possible to find a polynomial time algorithm for upward embedding testing of a given digraph on T_h.

- **Thank you for your attentions.**