# Fault-free Hamiltonian Cycles in Pancake Graphs with Conditional Edge Faults

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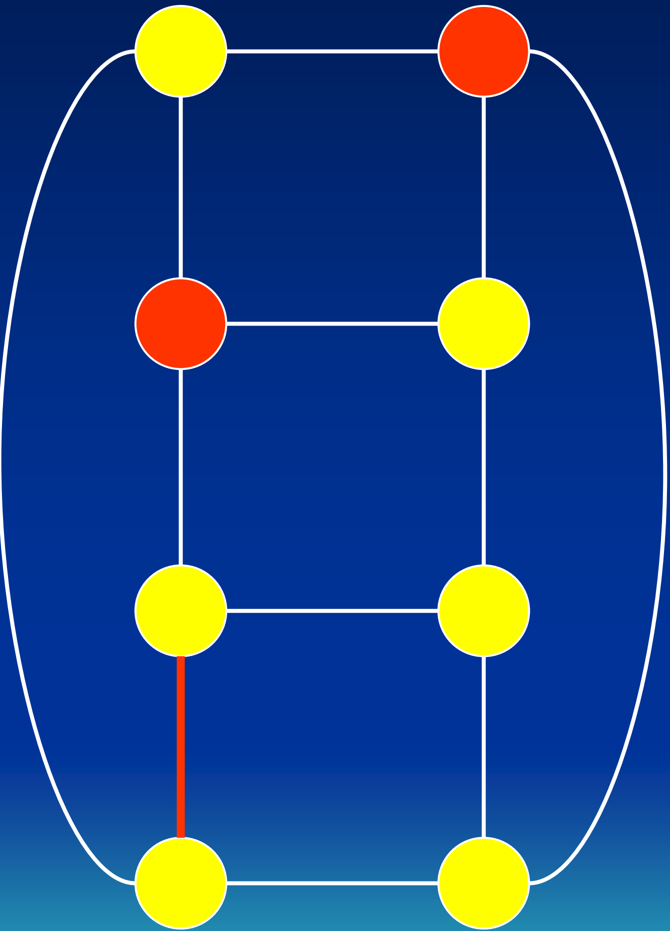
# Outline

- Hamiltonian problems
- Fault tolerant problems
- Pancake graphs
- Problem and previous results
- Main result and proof idea

# Hamiltonian Problems

- Ring (or linear array) embedding in networks
- Cycle (or path) embedding in graphs
- A cycle (or path) in a graph *G* is called a *Hamiltonian cycle* (or *Hamiltonian path* ) if it contains every vertex of *G* exactly once.
- *G* is called *Hamiltonian-connected* if every two vertices of *G* are connected by a Hamiltonian path.

# Fault Tolerant Problems

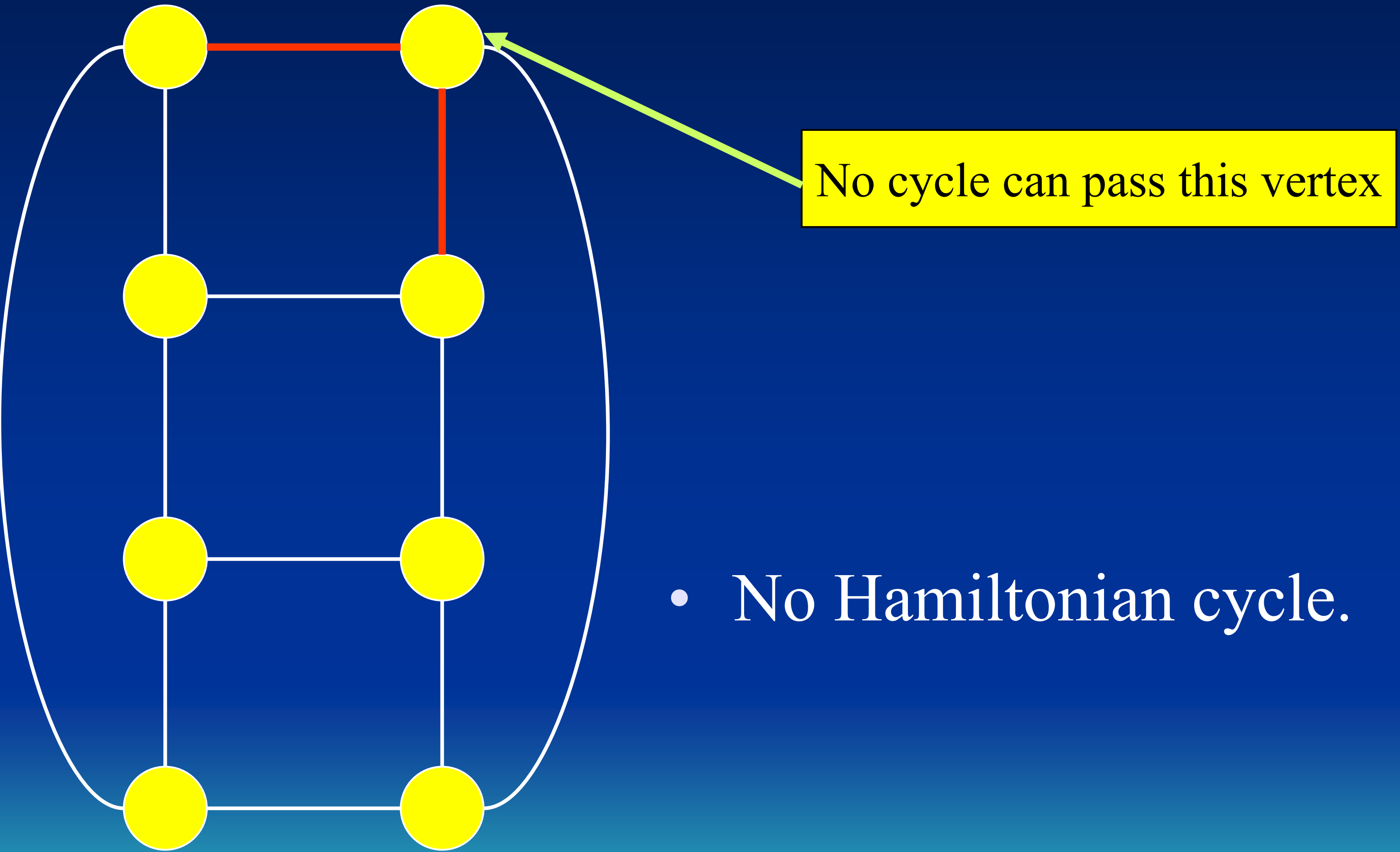


- Vertex faults (node faults).
- Edge faults (link faults).
- After some faults occur, does the network still work?

# Fault Models

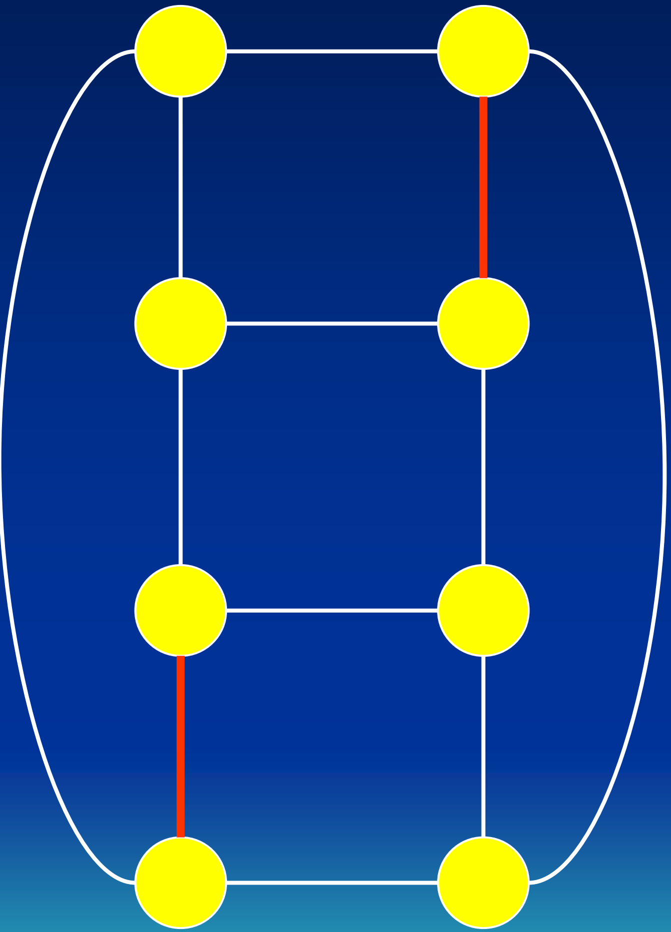
- *Random fault model*: assumed that the faults might occur everywhere without any restriction.
- *Conditional fault model*: assumed that the distribution of faults must satisfy some properties.
- It is more difficult to solve problems under the conditional fault model than the random fault model.

# Random Fault – Cycle Embedding




- No Hamiltonian cycle.

# Conditional Fault – Cycle Embedding



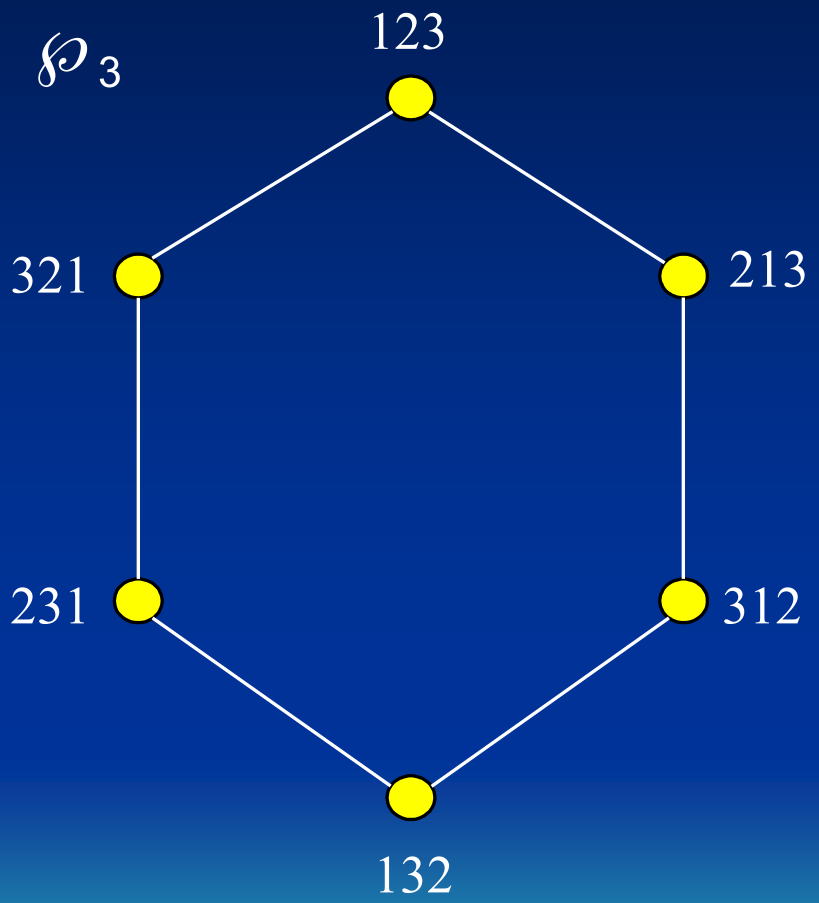
Each vertex has two healthy edges

- No Hamiltonian cycle.

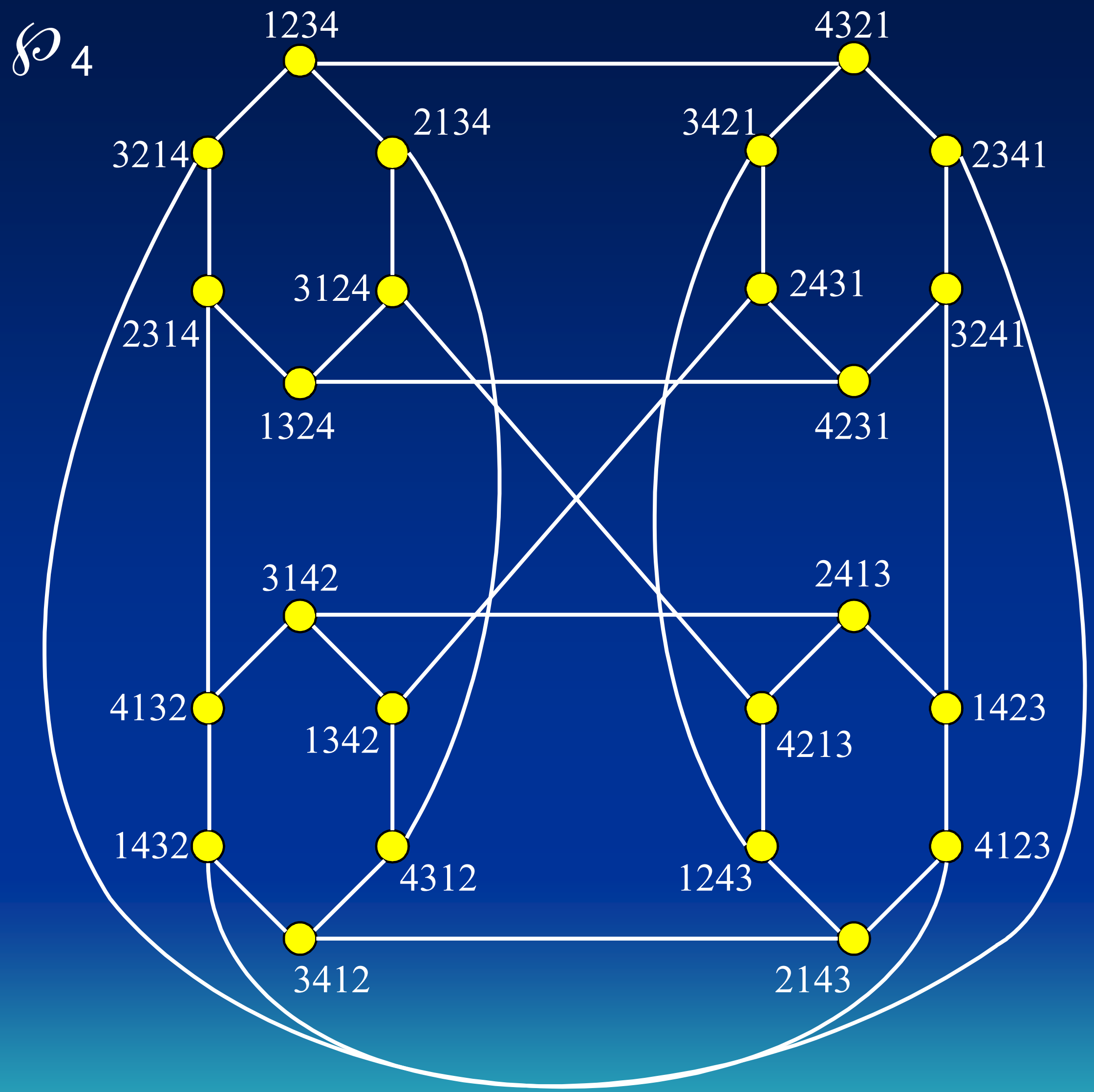
# Pancake Graphs

- A $n$-dimensional pancake graph, denoted by $\wp_n$, has the vertex set $V(\wp_n) = \{a_1a_2\ldots a_n \mid a_1a_2\ldots a_n$ is a permutation of 1, 2, …, $n\}$, and edge set $E(\wp_n) = (a_1a_2\cdots a_n, b_1b_2\cdots b_n) \mid a_1a_2\cdots a_k = b_kb_{k-1}\cdots b_1$ and $a_{k+1}a_{k+2}\cdots a_n = b_{k+1}b_{k+2}\cdots b_n$ for some $2 \le k \le n$ }.

$\wp_2$
12
21

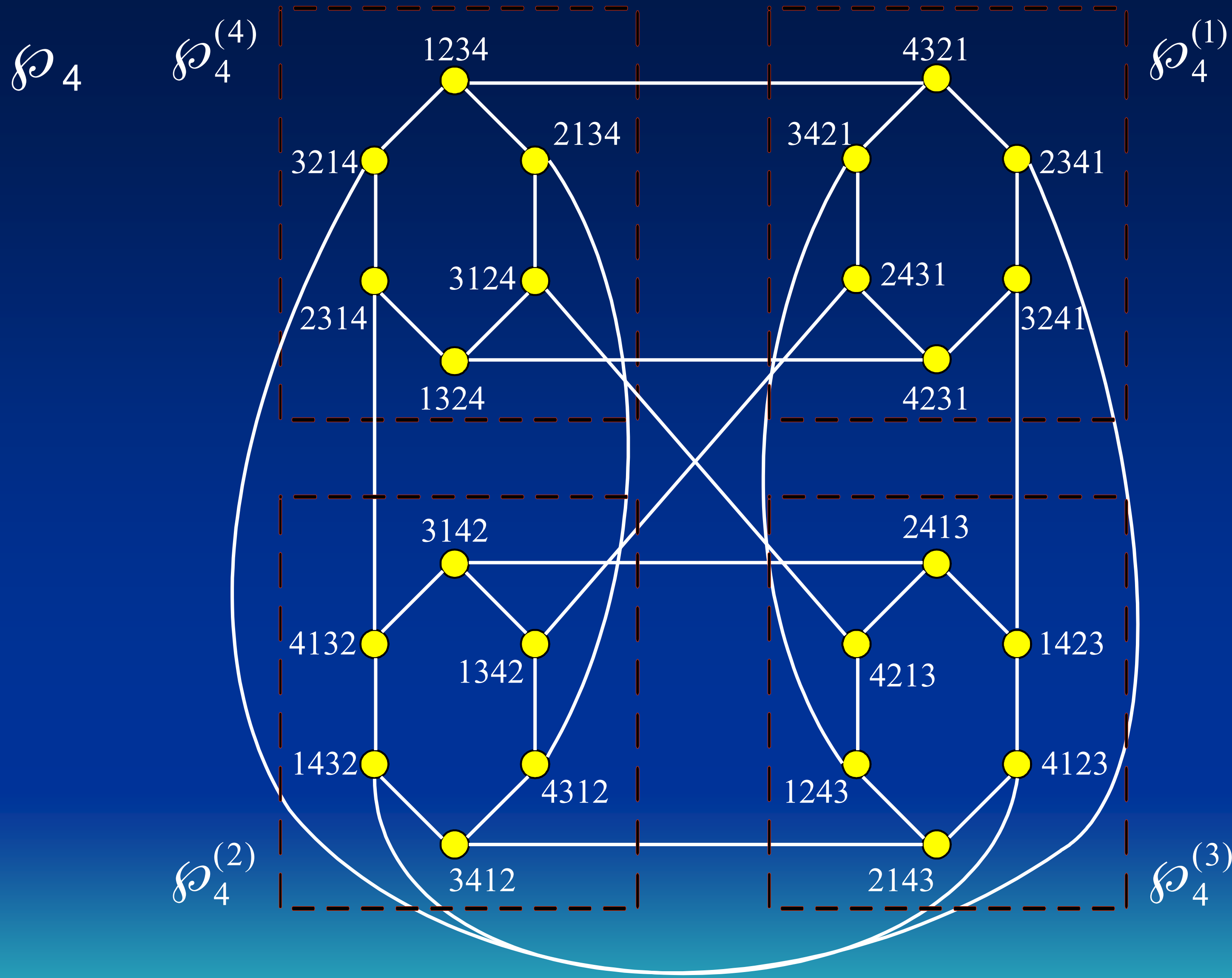
$\wp_3$
123
321
213
231
312
132



℘ 4
1234
4321
3214
2134
3421
2341
2314
3124
2431
3241
1324
4231
3142
2413
4132
1342
4213
1423
1432
4312
1243
4123
3412
2143

# Some Properties of $\wp_n$

- $\wp_n$ is regular of degree $n-1$.
- $\wp_n$ has $n!$ vertices and $(n-1)n!/2$ edges.
- The girth of $\wp_n$ is 6, where $n \geq 3$.
- $\wp_n$ belongs to the class of Cayley graphs.
- $\wp_n$ is vertex symmetric, but not edge symmetric.
- $\wp_n$ is Hamiltonian-connected.
- $\wp_n$ is a recursive structure.



℘ 4
℘4(4)
℘4(1)
℘4(2)
℘4(3)
1234
3214
2134
2314
3124
1324
4321
3421
2341
2431
3241
4231
3142
4132
1342
1432
4312
3412
2413
4213
1423
1243
4123
2143

# A famous problem of $\wp_n$

- The pancake graph is named from the famous “pancake problem” whose answer is exactly the diameter of the corresponding pancake graph [1].
- The diameter of $\wp_n$ is bounded above by $3(n + 1)/2$. It is still an open problem to compute the exact diameter of $\wp_n$ [2].

- [1] W. H. Gates and C. H. Papadimitriou, “Bounds for sorting by prefix reversal,” *Discrete Mathematics*, vol. 27, pp. 47-57, 1979.
- [2] M. H. Heydari and I. H. Sudborough, “On the diameter of the pancake network,” *Journal of Algorithms*, vol. 25, pp. 67-94, 1997.

# Our Problem

- Fault component: edge faults only.
- Fault model: conditional fault model.
- Assumption: each vertex has at least two non-faulty edges.
- How many edge faults can $\wp_n$ tolerate while retaining a fault-free Hamiltonian cycle? ($n \geq 4$)
- This is the first result on the fault tolerance of the pancake graph under the conditional fault model.

# Previous results on $\wp_n$

| property \ model | Random fault model | Conditional fault model |
|---|---|---|
| *k*-edge-fault-tolerant Hamiltonian | $k \le n - 3$ § | $k \le 2n - 7$ |
| *k*-edge-fault-tolerant Hamiltonian-connected | $k \le n - 4$ § | $k \le n - 4$ * |

§ C. N. Hung, H. C. Hsu, K. Y. Liang and L. H. Hsu, "Ring embedding in faulty pancake graphs," *Information Processing Letters*, vol. 86, pp. 271-275, 2003.

* P. Y. Tsai, "Edge-fault-tolerant path/cycle embedding on some Cayley graphs," Ph.D. Thesis, National Taiwan University, Taipei, Taiwan, to appear (2008).

# Lemmas

- Lemma 1: $|\tilde{E}_{p,q}(\wp_n)| = (n-2)!$ for all $p, q \in \{1, 2, \ldots, n\}$ and $p \neq q$, where $n \geq 3$.
- Lemma 2: $\wp_n - F$ is Hamiltonian if $|F| \leq n-3$, and Hamiltonian-connected if $|F| \leq n-4$, where $n \geq 4$ ($F$ denotes a set of edge faults in $\wp_n$) .

# Lemmas

- Lemma 3: Suppose that $u, v \in V(\wp_n)$ and $\langle u \rangle_n \neq \langle v \rangle_n$, where $n \geq 5$. For any $I \subseteq \{1, 2, \ldots, n\}$ and $|I| \geq 2$, there exists a Hamiltonian path from $u$ to $v$ in $\wp_n^I - F$ provided the following two conditions hold:

  (C1) $|\tilde{E}_{i,j}(\wp_n) - F| \geq 3$ for all $i, j \in I$ and $i \neq j$;

  (C2) $\wp_n^{(r)} - F$ is Hamiltonian-connected for all $r \in I$.
- Lemma 4: Suppose that $u, v \in V(\wp_n^{(r)})$ and $u \neq v$, where $r \in \{1, 2, \cdots, n\}$ and $n \geq 4$. If $d_{u,v} \leq 2$, then $\langle N^{(n)}(u) \rangle_n \neq \langle N^{(n)}(v) \rangle_n$, where $d_{u,v}$ is the distance between $u$ and $v$.

# Lemmas

- Lemma 5: Suppose that $e_1$, $e_2 \in E(\wp_4)$ and $e_1 \neq e_2$. There exists a Hamiltonian cycle in $\wp_4 - \{e_2\}$ that contains $e_1$.

- Lemma 6: Suppose that $s$, $t \in V(\wp_n)$, $s \neq t$, and $\langle s \rangle_1 = \langle t \rangle_1$, where $n \geq 4$. For every $(x, y) \in E(\wp_n)$ with $\{x, y\} \cap \{s, t\} = \varnothing$, there exists a Hamiltonian path from $s$ to $t$ in $\wp_n$ that contains $(x, y)$.

# Main Result

- Theorem: $\wp_n - F$ is Hamiltonian provided $|F| \leq 2n - 7$ and $\delta(\wp_n - F) \geq 2$ , where $n \geq 4$ ($F$ denotes a set of edge faults in $\wp_n$) .

# Proof idea

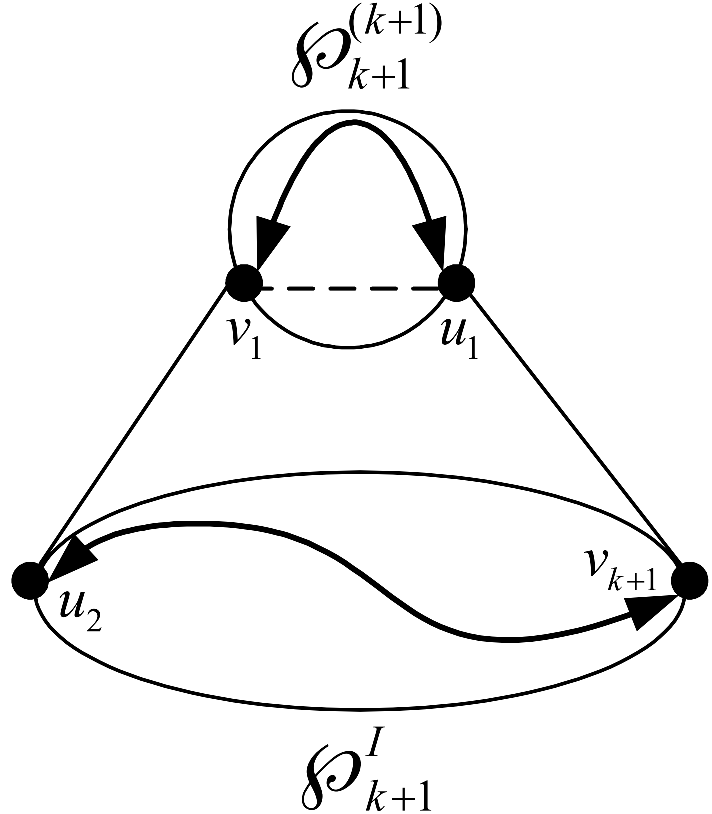
- The theorem holds for $\wp_4$, which is assured by Lemma 2 ($2n - 7 = n - 3$ as $n = 4$).
- Prove by induction on $n$.
- Suppose the theorem holds for $\wp_k$, now we construct a Hamiltonian cycle in $\wp_{k+1} - F$, where $k \geq 4$ and $|F| \leq 2k - 5$.

# Proof idea

- Assume that $|E(\wp_{k+1}^{(k+1)}) \cap F| \geq |E(\wp_{k+1}^{(k)}) \cap F| \geq \cdots \geq |E(\wp_{k+1}^{(1)}) \cap F|$ .
- By Lemma 1, we have $|\tilde{E}_{p,q}(\wp_{k+1})| = (k-1)! \geq 2k-2 \geq |F|+3$, i.e., $|\tilde{E}_{p,q}(\wp_{k+1}) - F| \geq 3$ for all $p, q \in \{1, 2, \cdots, k+1\}$ and $p \neq q$.
- Four cases are discussed.

# Case 1

- $|E(\wp_{k+1}^{(k+1)}) \cap F| \leq k-4$.
- The induction hypothesis assures a Hamiltonian cycle $C$ in $\wp_{k+1}^{(k+1)} - F$.
- An edge $(u_1, v_1)$ can be determined from $C$ so that there exist $(v_1, u_2)$, $(u_1, v_{k+1}) \in E^{(k+1)}(\wp_{k+1}) - F$ with $u_2, v_{k+1} \in V(\wp_{k+1}^{I})$, where $I = \{1, 2, \cdots, k\}$.
- Lemma 2 assures that $\wp_{k+1}^{(j)} - F$ is Hamiltonian-connected for all $1 \leq j \leq k$.
- By Lemma 3, a Hamiltonian path in $\wp_{k+1}^{I} - F$ exists.



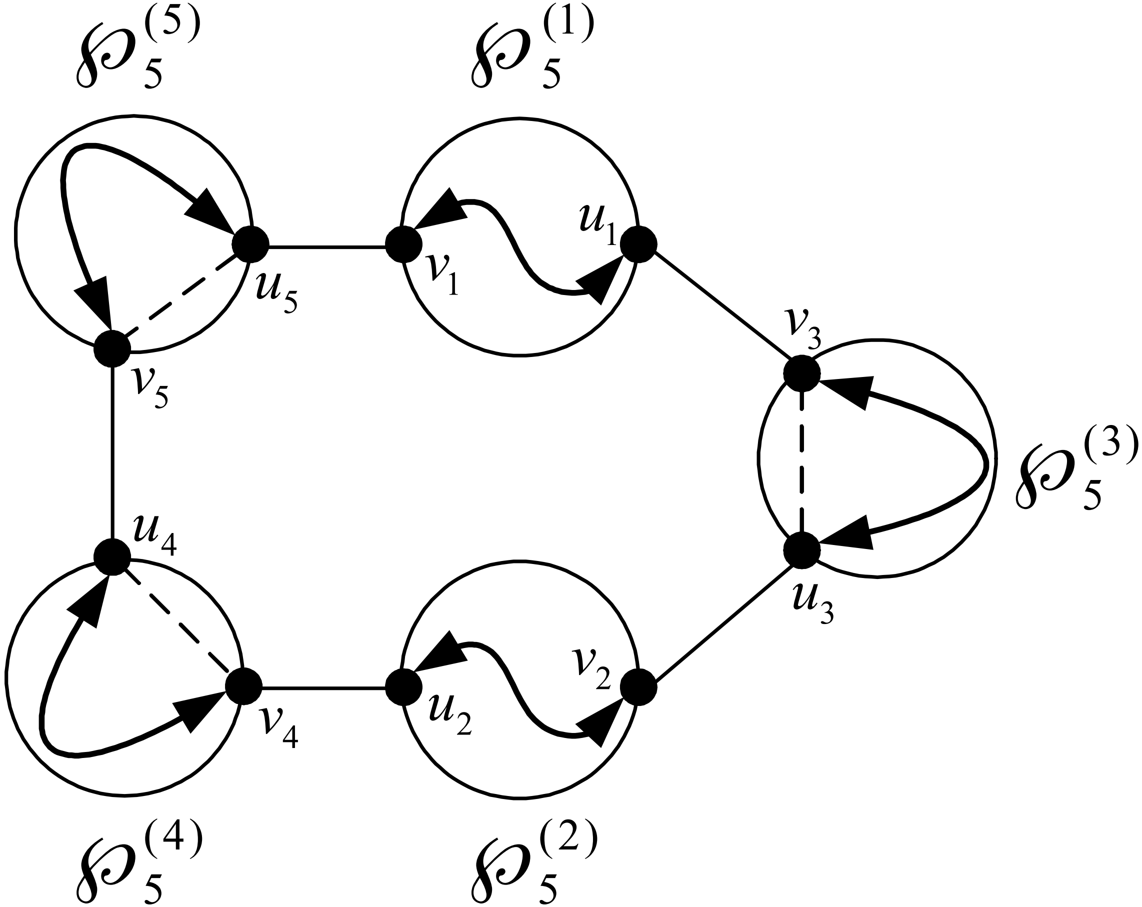
$\wp_{k+1}^{(k+1)}$
$v_1$
$u_1$
$u_2$
$v_{k+1}$
$\wp_{k+1}^{I}$

# Case 2

- $k-3 \leq |E(\wp_{k+1}^{(k+1)}) \cap F| \leq 2k-7.$
- Subcase 1: $|E(\wp_{k+1}^{(k)}) \cap F| \leq k-4.$
- If $\delta(\wp_{k+1}^{(k+1)} - F) \geq 2$, the Hamiltonian cycle can be obtained by the construction method of Case 1.
- If $\delta(\wp_{k+1}^{(k+1)} - F) = 1$, we can construct the Hamiltonian cycle by slightly modifying the construction method of Case 1 (use the same figure).

# Case 2(continue)

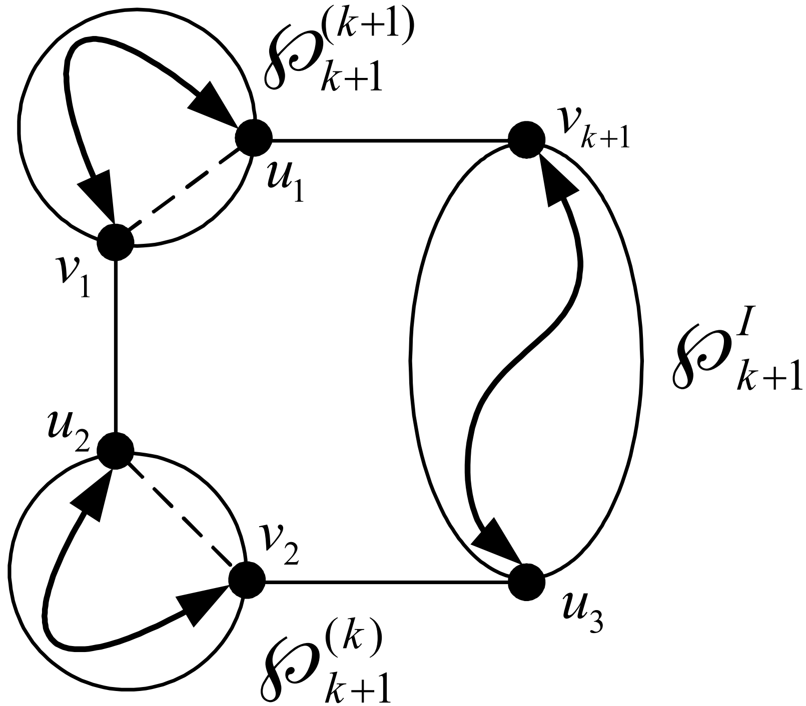
- Subcase 2: $|E(\wp_{k+1}^{(k)}) \cap F| \geq k-3$.
- We have $|E(\wp_{k+1}^{(k+1)}) \cap F| = k-3$ or $k-2$, $|E(\wp_{k+1}^{(k)}) \cap F| = k-3$ (hence, $\delta(\wp_{k+1}^{(k)} - F) \geq 2$), and $|E^{(k+1)}(\wp_{k+1}) \cap F| \leq 1$.
- First we consider the situation of $|E(\wp_{k+1}^{(k-1)}) \cap F| = k-3$, which occurs only when $k=4$.
- We have $|E(\wp_5^{(5)}) \cap F| = |E(\wp_5^{(4)}) \cap F| = |E(\wp_5^{(3)}) \cap F| = 1$, $|E(\wp_5^{(2)}) \cap F| = |E(\wp_5^{(1)}) \cap F| = 0$, and $|E^{(5)}(\wp_5) \cap F| = 0$.
- Assisted by Lemma 5.



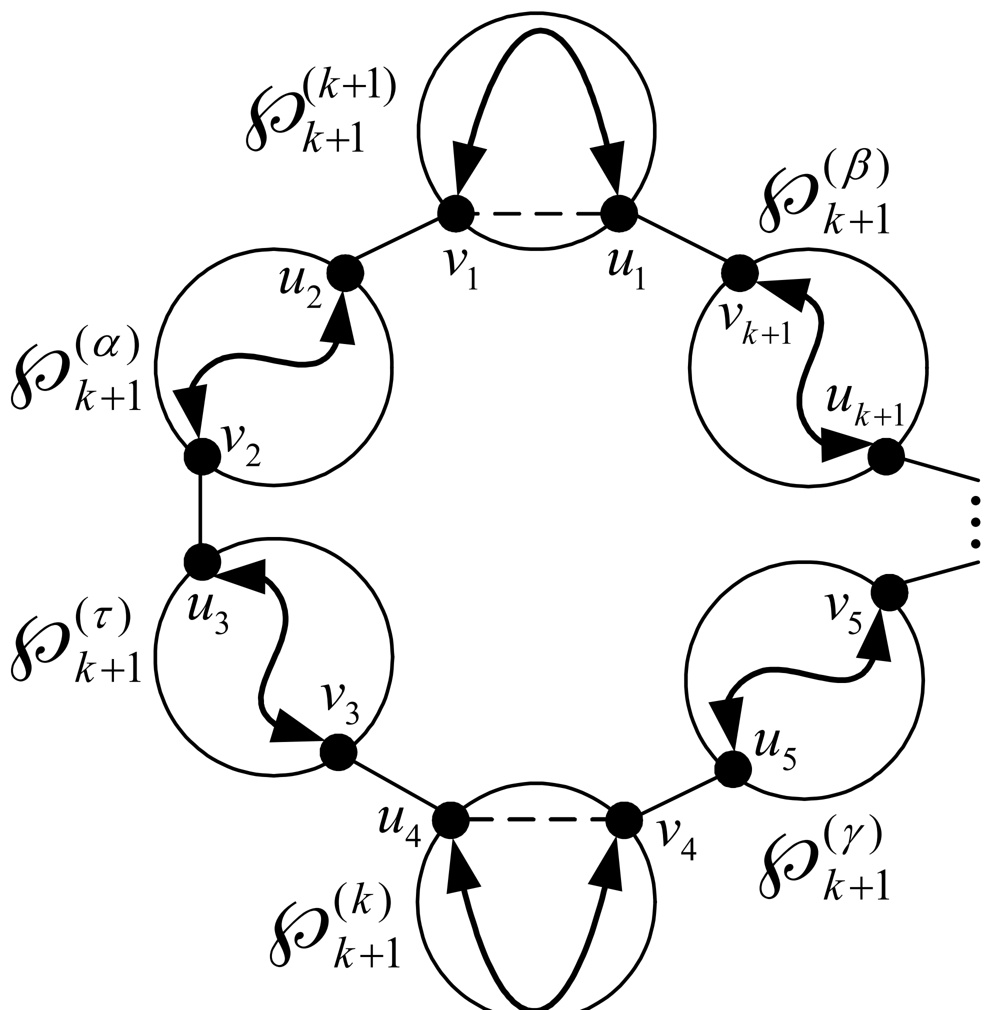
$\wp_5^{(5)}$
$\wp_5^{(1)}$
$\wp_5^{(3)}$
$\wp_5^{(4)}$
$\wp_5^{(2)}$
$u_1$
$v_1$
$u_2$
$v_2$
$u_3$
$v_3$
$u_4$
$v_4$
$u_5$
$v_5$

# Case 2(continue)

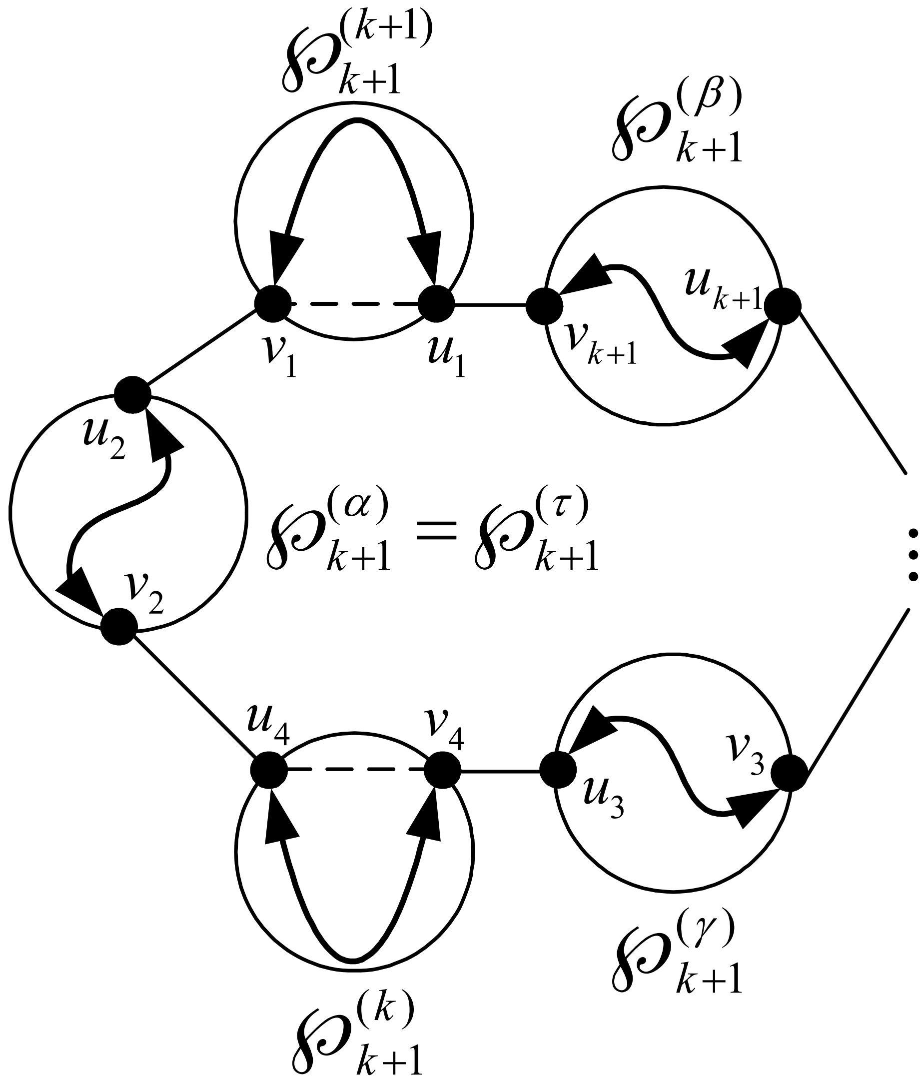
- Now we consider the situation of $|E(\wp_{k+1}^{(k-1)}) \cap F| \leq k-4$.
- When $\delta(\wp_{k+1}^{(k+1)} - F) \geq 2$.
- When $\delta(\wp_{k+1}^{(k+1)} - F) = 1$.



$\wp_{k+1}^{(k+1)}$
$u_1$
$v_{k+1}$
$v_1$
$\wp_{k+1}^{I}$
$u_2$
$v_2$
$u_3$
$\wp_{k+1}^{(k)}$



$\wp_{k+1}^{(k+1)}$
$v_1$
$u_1$
$\wp_{k+1}^{(\beta)}$
$u_2$
$v_{k+1}$
$\wp_{k+1}^{(\alpha)}$
$u_{k+1}$
$v_2$
$u_3$
$\wp_{k+1}^{(\tau)}$
$v_5$
$v_3$
$u_5$
$u_4$
$v_4$
$\wp_{k+1}^{(\gamma)}$
$\wp_{k+1}^{(k)}$



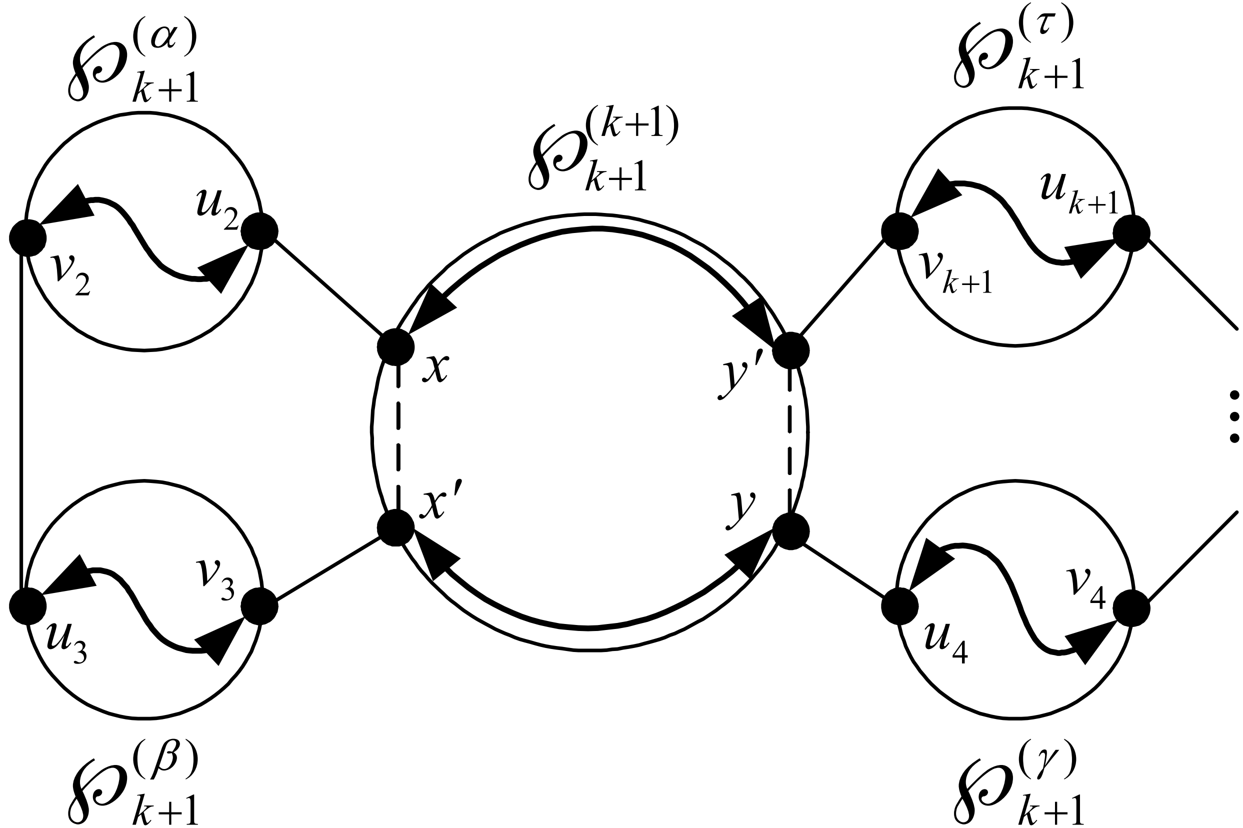
$\wp_{k+1}^{(k+1)}$
$\wp_{k+1}^{(\beta)}$
$u_{k+1}$
$v_1$
$u_1$
$v_{k+1}$
$u_2$
$\wp_{k+1}^{(\alpha)} = \wp_{k+1}^{(\tau)}$
$v_2$
$u_4$
$v_4$
$v_3$
$u_3$
$\wp_{k+1}^{(\gamma)}$
$\wp_{k+1}^{(k)}$

# Case 3

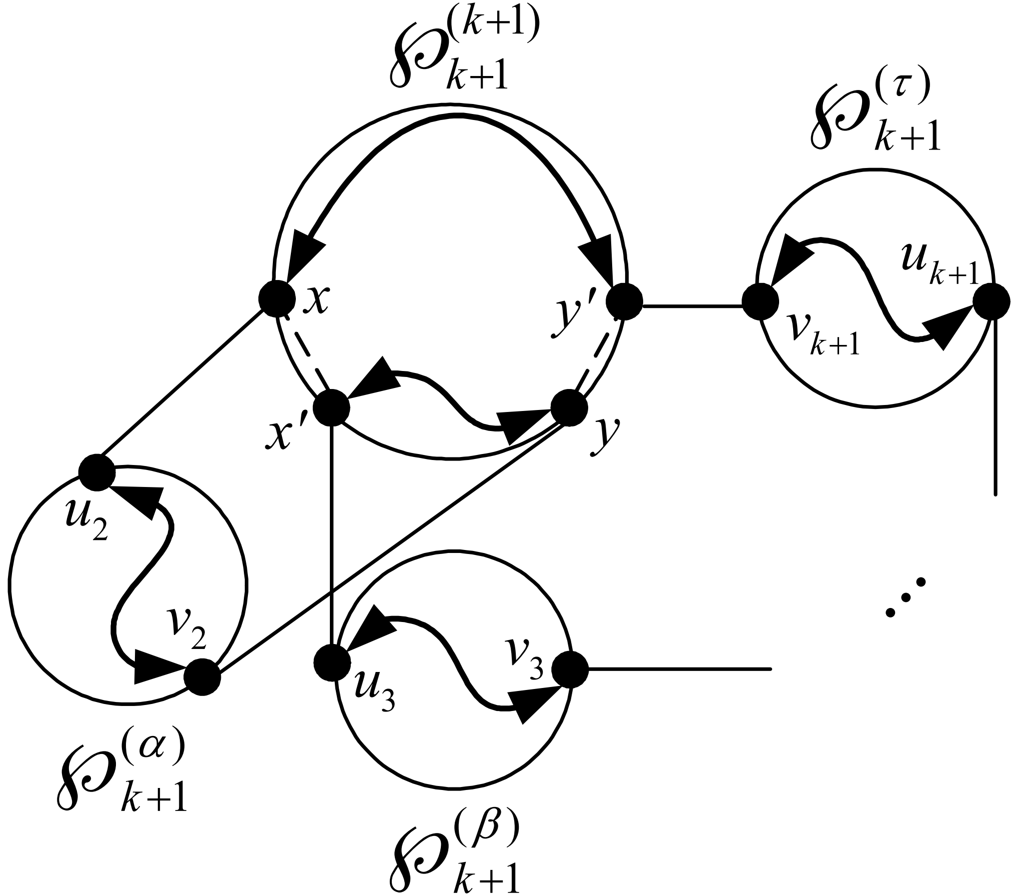
- $|E(\wp_{k+1}^{(k+1)}) \cap F| = 2k - 6.$
- We have $|E^{(k+1)}(\wp_{k+1}) \cap F| \leq 1$ and $|E(\wp_{k+1}^{(j)}) \cap F| \leq 1$ for all $1 \leq j \leq k$.
- When $k \geq 5$, the Hamiltonian cycle can be obtained by slightly modifying the construction method of Case 1.
- When $k = 4$, if $|E(\wp_5^{(j)}) \cap F| = 0$ for all $1 \leq j \leq 4$, the Hamiltonian cycle can be obtained all the same as the situation of $k \geq 5$. Otherwise, the Hamiltonian cycle can be obtained similar to the construction method of Case 2.

# Case 4

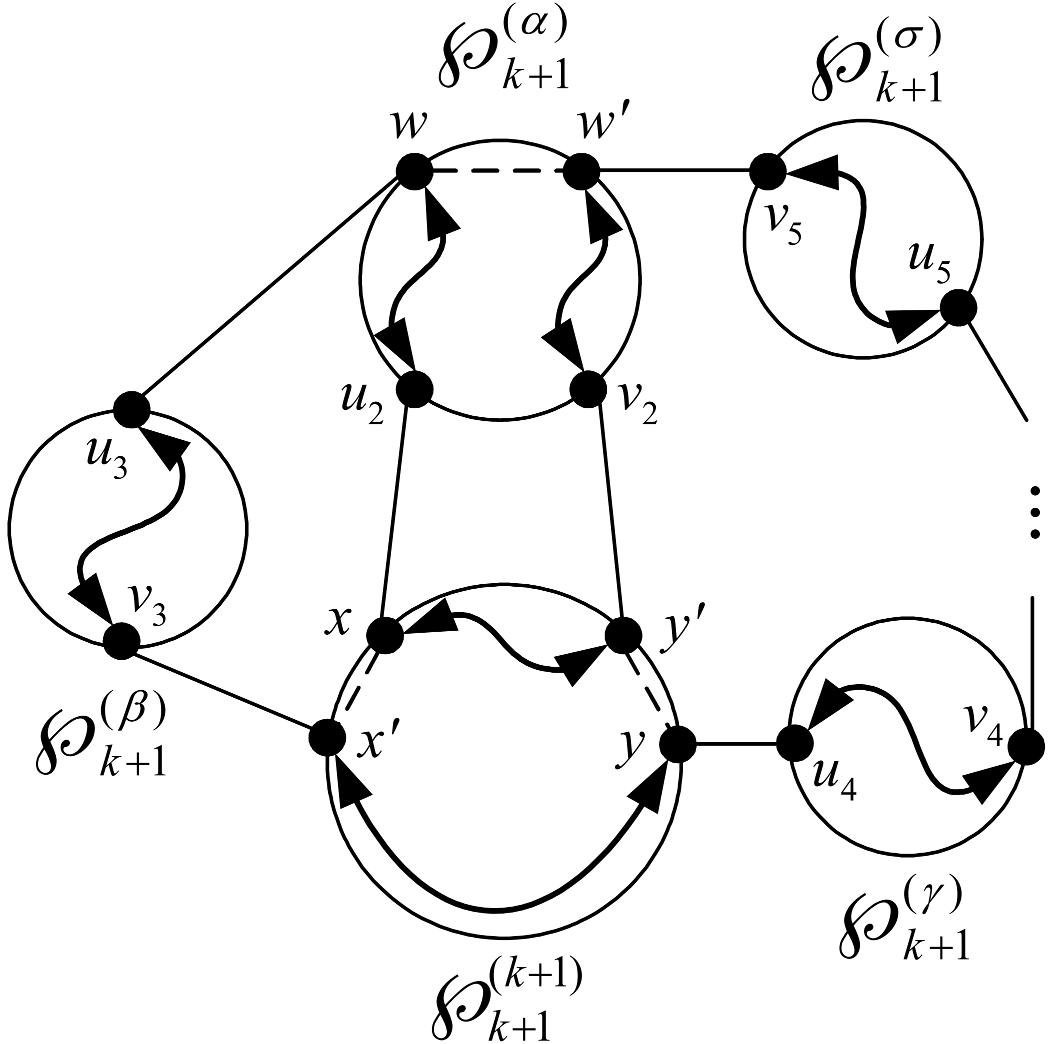
- $|E(\wp_{k+1}^{(k+1)}) \cap F| = 2k - 5$.
- First, two edges $(x, x')$, $(y, y') \in E(\wp_{k+1}^{(k+1)}) \cap F$ are determined so that $\{x, x'\} \cap \{y, y'\} = \emptyset$ and $\delta(\wp_{k+1}^{(k+1)} - (F - \{(x, x'), (y, y')\})) \geq 2$.
- The induction hypothesis assures a Hamiltonian cycle $C$ in $\wp_{k+1}^{(k+1)} - (F - \{(x, x'), (y, y')\})$.
- If $(x, x')$ or $(y, y')$ is not contained in $C$, the Hamiltonian cycle can be obtained by the construction method of Case 1.
- Otherwise, five figures are considered.



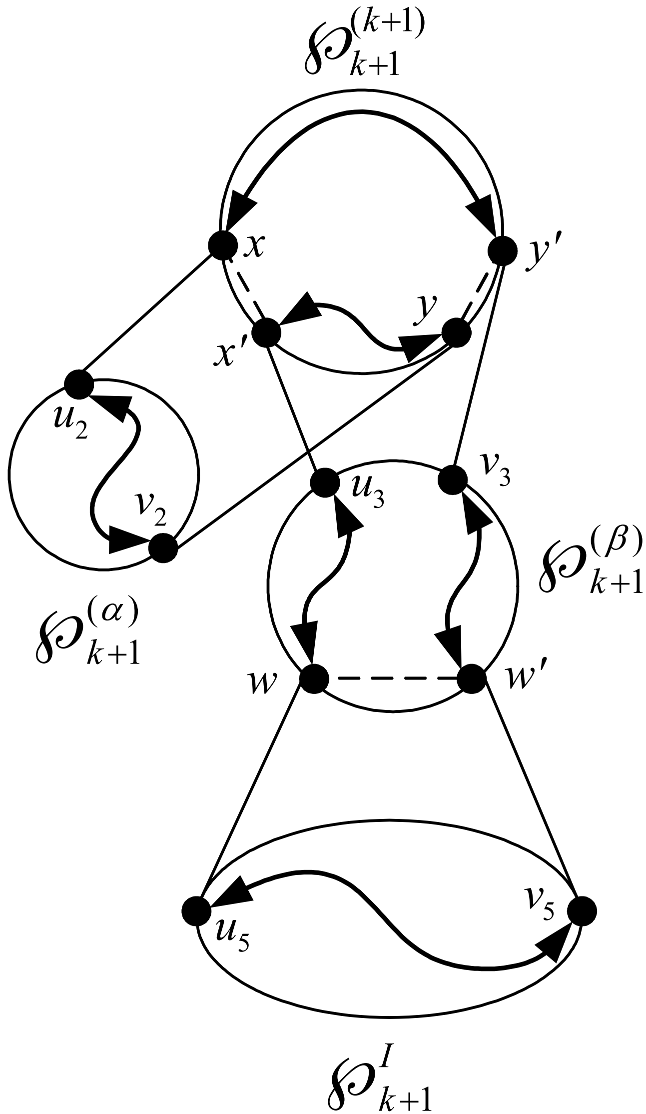
$\wp_{k+1}^{(\alpha)}$
$\wp_{k+1}^{(k+1)}$
$\wp_{k+1}^{(\tau)}$
$u_2$
$v_2$
$x$
$y'$
$u_{k+1}$
$v_{k+1}$
$x'$
$y$
$v_3$
$u_3$
$u_4$
$v_4$
$\wp_{k+1}^{(\beta)}$
$\wp_{k+1}^{(\gamma)}$



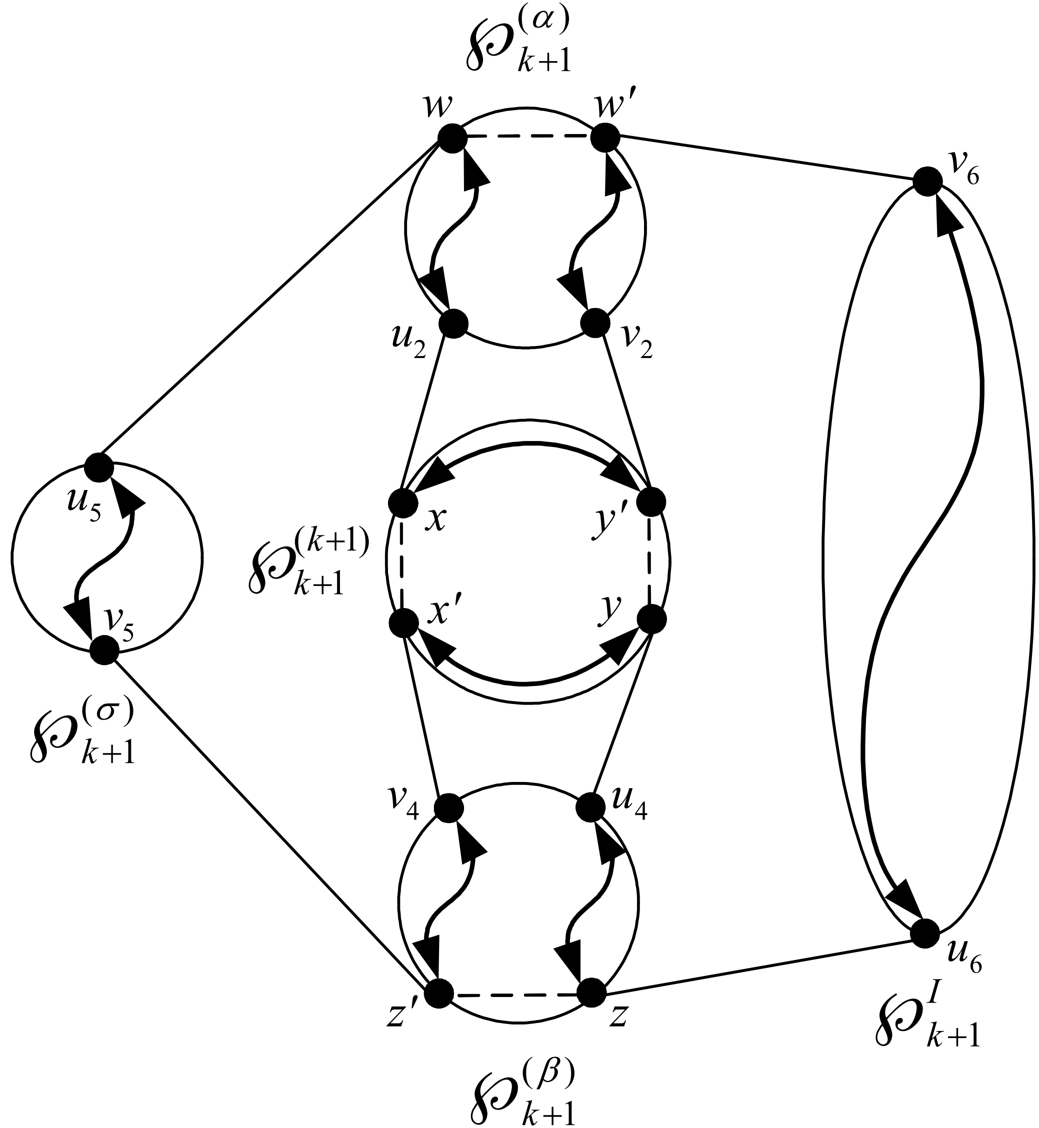
$\wp_{k+1}^{(k+1)}$
$\wp_{k+1}^{(\tau)}$
$x$
$y'$
$x'$
$y$
$u_{k+1}$
$v_{k+1}$
$u_2$
$v_2$
$u_3$
$v_3$
$\wp_{k+1}^{(\alpha)}$
$\wp_{k+1}^{(\beta)}$



$\wp_{k+1}^{(\alpha)}$
$\wp_{k+1}^{(\sigma)}$
$w$
$w'$
$v_5$
$u_5$
$u_2$
$v_2$
$u_3$
$v_3$
$x$
$y'$
$x'$
$y$
$u_4$
$v_4$
$\wp_{k+1}^{(\beta)}$
$\wp_{k+1}^{(\gamma)}$
$\wp_{k+1}^{(k+1)}$

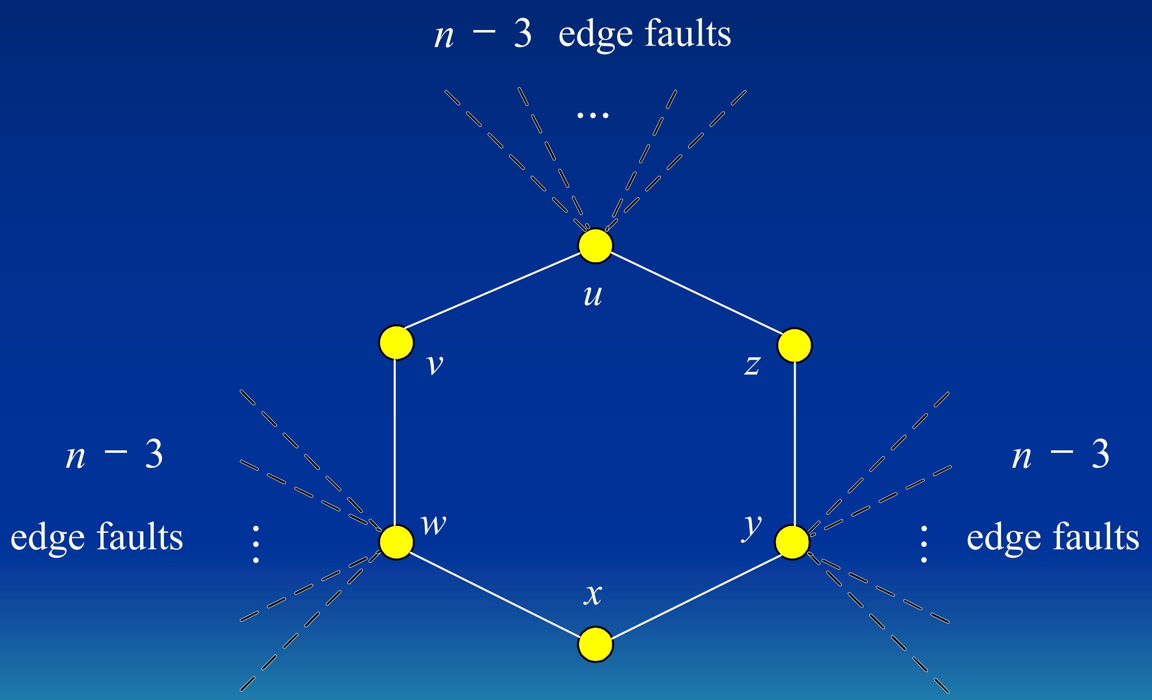


$\wp_{k+1}^{(k+1)}$
$x$
$y'$
$y$
$x'$
$u_2$
$v_2$
$u_3$
$v_3$
$\wp_{k+1}^{(\beta)}$
$\wp_{k+1}^{(\alpha)}$
$w$
$w'$
$v_5$
$u_5$
$\wp_{k+1}^{I}$



$\wp_{k+1}^{(\alpha)}$
$w$
$w'$
$v_6$
$u_2$
$v_2$
$u_5$
$x$
$y'$
$\wp_{k+1}^{(k+1)}$
$x'$
$y$
$v_5$
$\wp_{k+1}^{(\sigma)}$
$v_4$
$u_4$
$u_6$
$z'$
$z$
$\wp_{k+1}^{I}$
$\wp_{k+1}^{(\beta)}$

A distribution of $3n - 9$ edge faults over an $n$-dimensional pancake graph. No fault-free Hamiltonian cycle can be found for this situation.



# Open problem

- There is an upper bound of $3n - 10$ on the greatest number of tolerable edge faults for the problem.
- It is an open problem to narrow down the gap between $2n - 7$ and $3n - 10$.