Improving the gap of Erdős-Pósa property for minor-closed graph classes

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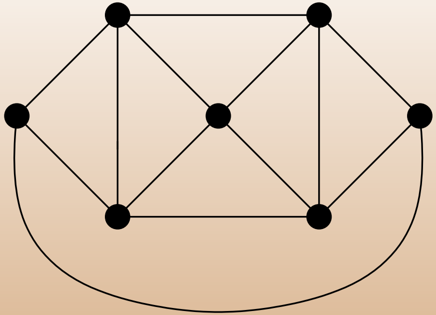
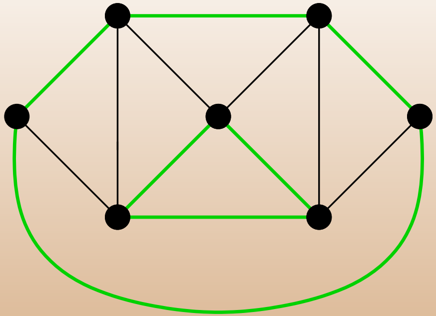
Given a graph G,

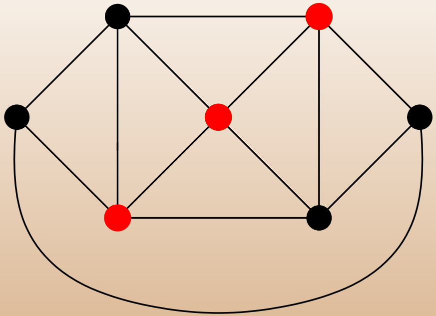
Cycle packing number: $\mathbf{cp}(G) =$ max # of disjoint cycles in G

Feedback vertex set: $\mathbf{fvs}(G) =$ min # of vertices covering all cycles in G

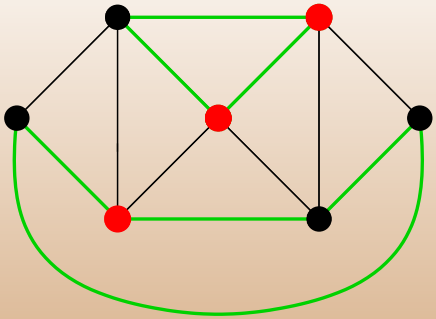




$\mathbf{cp}(G) = 2$



$\mathbf{fvs}(G) = 3$



$\textbf{fvs}(G) = 3$ $\textbf{cp}(G) = 2$

Theorem (Erdős-Pósa)

There is some function f such that for any graph G,

$\mathbf{cp}(G) \leq \mathbf{vfs}(G) \leq f(\mathbf{cp}(G))$.

[Paul Erdős and Luis Pósa. On independent circuits contained in a graph. *Canad. J. Math.*, 17:347–352, 1965.]

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Here, $f(k) = O(k \cdot \log k)$

Let $\mathcal{H}$ be a graph class.

$$\textbf{cover}_{\mathcal{H}}(G) = \min\{k \mid \exists\ S \subseteq V(G)\ \forall_{H \in \mathcal{H}} H \not\subseteq G \setminus S\}.$$

$$\textbf{pack}_{\mathcal{H}}(G) = \max\{k \mid \exists \text{ a partition } V_1, \ldots, V_k \text{ of } V(G)$$

$$\text{such that } \forall_{i \in \{1,\ldots,k\}} \exists_{H \in \mathcal{H}}\ H \subseteq G[V_i]\}.$$

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$\mathcal{H}$ *has the Erdős-Pósa property* for $\mathcal{G}$ if there is a function f (depending only on $\mathcal{H}$ and $\mathcal{G}$) such that, for any graph $G \in \mathcal{G}$,

$$\textbf{pack}_{\mathcal{H}}(G) \leq \textbf{cover}_{\mathcal{H}}(G) \leq f(\textbf{pack}_{\mathcal{H}}(G))$$

Some notation:

$H \leq_c G$ (H *is a contraction* G) if H can be obtained from G after a series of edge contractions

$H \leq_m G$ (H *is a minor of* G) if some subgraph of G can be contracted to H.

A graph class $\mathcal{G}$ is *minor-closed* if any minor of a graph in $\mathcal{G}$ is again a member of $\mathcal{G}$ (e.g. planar graphs).

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What about other choices of H ?

Proposition (**12.4.10** in Diestel's Book on Graph Theory)

Let H be a connected graph. Then $\mathcal{M}(H)$ satisfies the Erdős-Pósa property for all graphs if and only if H is planar.

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Question: What about if G belongs in some sparse graph class?

Theorem

Let H be a connected planar graph and let $\mathcal{G}$ be a non-trivial minor closed graph class. Then $\mathcal{M}(H)$ satisfies the Erdős-Pósa property for $\mathcal{G}$ with a **linear** *gap function f.*

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i.e. there is a constant $\sigma_{\mathcal{G},H}$ such that, for any $G \in \mathcal{G}$,

$$\mathbf{pack}_{\mathcal{M}(H)}(G) \leq \mathbf{cover}_{\mathcal{M}(H)}(G) \leq \sigma_{\mathcal{G},H} \cdot \mathbf{pack}_{\mathcal{M}(H)}(G)$$

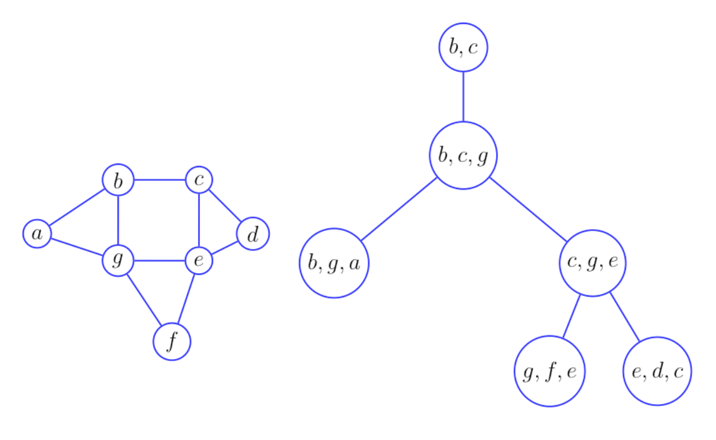
A *tree decomposition* of a graph G is a pair $D = (T, \mathcal{X})$ such that T is a tree and $\mathcal{X} = \{X_t \mid t \in V(T)\}$ is a collection of subsets of G.

(each $X_t \in \mathcal{X}$ corresponds to a vertex $t \in V(T)$ – we call X_t *node* of D)

such that the following conditions are satisfied:

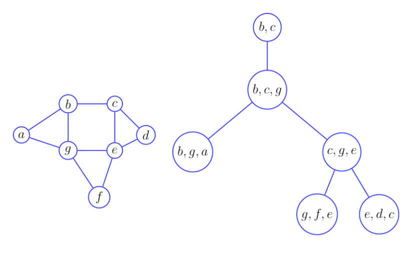
- Any vertex $v \in V(G)$ and the endpoints of any edge $e \in E(G)$ belong in some node X_t of D
- For any $v \in V(G)$, the set $\{t \in V(T) \mid v \in X_t\}$ is a subtree of T.

Tree Decompositions



The *width* of a tree decomposition $(T, \mathcal{X})$ is $\max_{t \in V(T)} |X_t| - 1$

The *tree-width* of a graph G (**tw**(G)) is the minimum width over all tree decompositions of G

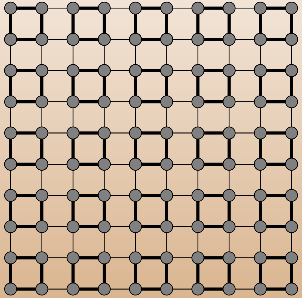


Lemma

If H is a planar graph H and $\mathcal{G}$ is a non-trivial minor-closed graph class, then, there is a constant $c_{\mathcal{G},H}$, depending only on $\mathcal{G}$ and H such that for any graph $G \in \mathcal{G}$,

$\mathbf{tw}(G) \leq c_{\mathcal{G},H} \cdot (\mathbf{pack}_{\mathcal{M}(H)}(G))^{1/2}$.

If $H = K_3$,



Then $\mathbf{pack}_{\mathcal{M}(K_3)}(G) \leq k$

implies the exclusion of a $(O(\sqrt{k}) \times O(\sqrt{k}))$-grid as a a minor

which in turn implies a $O(\sqrt{k})$ bound for **tw**(G).

Given a graph G, we call a triple (V_1, S, V_2) d-*separation triple* of G if $|S| \leq d$ and $\{V_1, S, V_2\}$ is a partition of $V(G)$ such that there is no edge in G between a vertex in V_1 and a vertex in V_2.

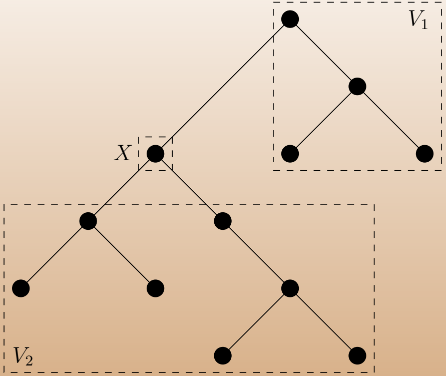
Using the tree structure of the decomposition we prove the following

Lemma

If H be a planar graph H and $\mathcal{G}$ is a non-trivial minor-closed graph class then for every $G \in \mathcal{G}$ where $\mathbf{pack}_{\mathcal{M}(H)}(G) = k$ there is an $c_{\mathcal{G},H} \cdot \sqrt{k}$-separation triple (V_1, X, V_2) of G, where

$k/3 \leq \mathbf{pack}_{\mathcal{M}(H)}(G[V_1]) \leq 2k/3$ *and*

$\mathbf{pack}_{\mathcal{M}(H)}(G[V_1]) + \mathbf{pack}_{\mathcal{M}(H)}(G[V_2]) \leq \mathbf{pack}_{\mathcal{M}(H)}(G)$



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Using now the fact that

$\textbf{cover}_{\mathcal{M}(H)}(G) \leq \textbf{cover}_{\mathcal{M}(H)}(G[V_1]) + \textbf{cover}_{\mathcal{M}(H)}(G[V_2]) + c_{\mathcal{G},H} \cdot \sqrt{k}$

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We can build an inductive argument that yields

$$\textbf{cover}_{\mathcal{M}(H)}(G) = O(\textbf{pack}_{\mathcal{M}(H)}(G)).$$

We proved that if H is planar and connected and $\mathcal{G}$ is non-trivial minor closed, then for any $G \in \mathcal{G}$,

$$\textbf{pack}_{\mathcal{M}(H)}(G) \leq \textbf{cover}_{\mathcal{M}(H)}(G) \leq \sigma_{\mathcal{G},H} \cdot \textbf{pack}_{\mathcal{M}(H)}(G)$$

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- Are there other, more wide, graph classes where a linear (or at least polynomial) gap can be detected?
- Are there algorithmic consequences of the linear gap?
- What about the constants $\sigma_{\mathcal{G},H}$ in the linear gap?

Cena in Emmaus, Michelangelo Merisi da Caravaggio, 1602, olio su tela, 141 $\times$ 196,2 cm. Londra, National Gallery

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