A Note on LP Relaxations for the 1D Cutting Stock Problem with Setup Costs

Alessandro Aloisio, Claudio Arbib
{aloisio, arbib}@di.univaq.it
Dipartimento di Informatica
Università degli Studi di L’Aquila
L’Aquila, Italy

Fabrizio Marinelli
marinelli@diiga.univpm.it
D.I.I.G.A.
Università Politecnica delle Marche
Ancona, Italy

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Outline of the talk

- The cutting stock problem with setup costs
- A compact mathematical model
- Reformulations by discretization
- Reformulations’ lower bound analysis
- The role of damned big $M$
- Some computational results
The 1D Cutting Stock Problem (CSP)

Input

- an unlimited number of identical stock items (e.g. steel bars) of given length \( w \);
- a set \( I \) of 1D part-types. For each \( i \in I \), let \( w_i \leq w \) and \( d_i > 0 \) be respectively the length and the demand of part-type \( i \);

Objective

- Satisfy the demand of part types minimizing the number \( z^* \) of used stock items

Output

- A set of cutting patterns and the number of times each pattern is replicated (activation levels).

The Pattern Minimization Problem (PMP)

Objective

- Given \( z^* \), minimize the number of different cutting patterns that are used (the number of cutting machine setups)
A mathematical model for PMP

The compact quadratic integer programming formulation (Vanderbeck, 1999) is the natural extension of the assignment formulation for CSP (Kantorovich, 1960).

Parameters:
\[ w \]
length of stock items

\[ (w_i, d_i) \]
lengths and demands of part-types \((i \in I)\)

\[ z^* \]
value of an optimal CSP solution

Binary variables:

\[ y_j = \begin{cases} 
1 & \text{if cutting pattern } j \text{ is used at least once} \\
0 & \text{otherwise} 
\end{cases} \]

Integer variables:

\[ x_{ij} \]
number of items of part-type \(i\) cut from a stock item when cutting pattern \(j\) is used

\[ z_j \]
activation level of cutting pattern \(j\)
A mathematical model for PMP

Minimizing the number of distinct cutting patterns:

$$\min \sum_{j=1}^{z^*} y_j$$

Demand satisfaction of part type $i$ (non-linear):

$$\sum_{j=1}^{z^*} z_j x_{ij} = d_i \quad i \in I$$

Feasibility of cutting pattern $j$:

$$\sum_{i \in I} w_i x_{ij} \leq w y_j \quad j = 1, \ldots, z^*$$

Upper bound on the number of used stock items:

$$\sum_{j=1}^{z^*} z_j \leq z^*$$

Activation of cutting pattern $j$:

$$z_j \leq z^* y_j \quad j = 1, \ldots, z^*$$

The formulation exhibits some symmetry.

The linearization leads to a linear integer formulation having a weak linear relaxation.
Decomposition of integer programs

\[ \min c^T x \]

\[ Ax \geq b \]

\[ x \in X \]

where

\[ X = P \cap \mathbb{Z}_+ \]

\[ P \subseteq \mathbb{R}^n \text{ is a polytope} \]

Discretization is an integer analogue to the Dantzig-Wolfe decomposition principle for linear programming.

\[ X = \{p_1, \ldots, p_q\} \]

\[ X = \left\{ x \in \mathbb{R}^n \left| x = \sum_{i=1}^{q} \lambda_i p_i, \sum_{i=1}^{q} \lambda_i = 1, \lambda \in \{0,1\}^q \right. \right\} \]

- If \( \text{conv}(X) \subset P \) the decomposition provides a reformulation whose linear programming relaxation yields a tight lower bound.
- Substitution for \( x \) yields an integer program whose linear relaxation is typically solved by column generation
Reformulation [VAN]

\[ \min \sum_{j=1}^{z^*} y_j \]
\[ \sum_{j=1}^{z^*} z_j x_{ij} = d_i \quad i \in I \]
\[ \sum_{i \in I} w_i x_{ij} \leq w y_j \quad j = 1, \ldots, z^* \]
\[ \sum_{j=1}^{z^*} z_j \leq z^* \]
\[ z_j \leq z^* y_j \quad j = 1, \ldots, z^* \]

Parameters:
- \( K \) set of all the feasible cutting patterns
- \( a_{ik} \) number of parts of type \( i \) yielded by pattern \( k \)
- \( u_k \) upper bound to the activation level of pattern \( k \)

\[ u_k = \min_{i \in I, a_{ik} > 0} \left\{ z^*, \left\lceil \frac{d_i}{a_{ik}} \right\rceil \right\} \]

Variables:
- \( \lambda_{kx} = 1 \) iff pattern \( k \) is applied \( x \) times in the solution

Polytope \( P \)

\[ \min \sum_{k \in K} \sum_{x=1}^{u_k} \lambda_{kx} \]
\[ \sum_{k \in K} \sum_{x=1}^{u_k} x a_{ik} \lambda_{kx} = d_i \quad i \in I \]
\[ \sum_{k \in K} \sum_{x=1}^{u_k} x \lambda_{kx} \leq z^* \]
\[ \lambda_{kx} \in \{0,1\} \quad k \in K, x = 1, \ldots, u_k \]
Reformulation [VAN]: Pricing Problem

\[ \min \sum_{k \in K} \sum_{x=1}^{u_k} \lambda_{kx} \]
\[ \mu : \sum_{k \in K} \sum_{x=1}^{u_k} x a_{ik} \lambda_{kx} = d_i \quad i \in I \]
\[ \sigma : \sum_{k \in K} \sum_{x=1}^{u_k} x \lambda_{kx} \leq z^* \]
\[ \lambda_{kx} \in \{0,1\} \quad k \in K, x = 1,\ldots,u_k \]

For fixed \( x \) the pricing problem is a bounded integer knapsack problem.

The integer non-linear pricing problem can be solved by considering a pseudo-polynomial number of knapsack problems.
Reformulation [GG]

\[
\begin{align*}
\min & \sum_{j=1}^{z^*} y_j \\
\sum_{j=1}^{z^*} z_j x_{ij} &= d_i \quad i \in I \\
\sum_{i \in I} w_i x_{ij} &\leq w_j \quad j = 1, \ldots, z^* \\
\sum_{j=1}^{z^*} z_j &\leq z^* \\
z_j &\leq z^* y_j \quad j = 1, \ldots, z^*
\end{align*}
\]

**Polytope** \( P \)

\[
\begin{align*}
\min & \sum_{k \in K} \lambda_k \\
\sum_{k \in K} a_{ik} x_k &= d_i \quad i \in I \\
\sum_{k \in K} x_k &\leq z^* \\
x_k &\leq u_k \lambda_k \quad k \in K \\
x_k &\in \mathbb{N}, \lambda_k \in \{0,1\} \quad k \in K
\end{align*}
\]

**Parameters**
- \( K \): set of all the feasible cutting patterns
- \( a_{ik} \): number of parts of type \( i \) yielded by pattern \( k \)
- \( u_k \): upper bound to the activation level of pattern \( k \)

**Variables**
- \( x_k \): activation level of pattern \( k \)
- \( \lambda_k = 1 \) if pattern \( k \) is applied at least once
Reformulation [GG]: Pricing Problem

\[
\begin{align*}
\min & \quad \sum_{k \in K} \lambda_k \\
\text{subject to} & \quad \mu : \quad \sum_{k \in K} a_{ik} x_k = d_i, \quad i \in I \\
& \quad \sigma : \quad \sum_{k \in K} x_k \leq z^* \\
& \quad \gamma : \quad x_k \leq u_k \lambda_k, \quad k \in K \\
& \quad x_k \in \mathbb{N}, \lambda_k \in \{0,1\}, \quad k \in K
\end{align*}
\]

Dual prices

Primal variables \((x_k, \lambda_k)\) correspond to dual inequalities

\[
\begin{align*}
\sum_{i \in I} a_{ik} \mu_i - \sigma - \gamma_k & \leq 0 \\
\gamma_k u_k & \leq 1
\end{align*}
\]

\((\mu, \sigma, \gamma)\) is dual feasible if

\[
1 - u_k \left( \sum_{i \in I} a_{ik} \mu_i - \sigma \right) \geq 0
\]

A profitable column \((a_k, u_k)\) is a cutting pattern satisfying

\[
1 - u_k \left( \sum_{i \in I} a_{ik} \mu_i - \sigma \right) < 0
\]
### [VAN] vs. [GG]

<table>
<thead>
<tr>
<th></th>
<th>[VAN]</th>
<th>[GG]</th>
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</thead>
<tbody>
<tr>
<td># of constraints</td>
<td>$</td>
<td>I</td>
</tr>
<tr>
<td># of variables</td>
<td>$O(z^* \cdot 2^{</td>
<td>I</td>
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<tr>
<td>Pricing problem</td>
<td>The <strong>same</strong> non-linear integer program</td>
<td></td>
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<tr>
<td>Column generation</td>
<td>Pricing can be directly embedded in a column generation scheme</td>
<td>Column-and-row generation scheme is required</td>
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On lower bounds by linear relaxation

\[[\text{VAN}^{\text{LP}}], [\text{GG}^{\text{LP}}]:\] linear relaxation of [\text{VAN}] and [\text{GG}]

\[z_{V}^{\text{LP}}(u), z_{GG}^{\text{LP}}(u):\] optimal value of programs [\text{VAN}^{\text{LP}}] and [\text{GG}^{\text{LP}}]

**Observation**

[\text{VAN}] and [\text{GG}] can be obtained by mere dualization of [\text{KAN}] only if \( u_k = z^* \). In this case Lagrangian theory tells us that formulation [\text{GG}^{\text{LP}}] cannot be stronger than [\text{VAN}^{\text{LP}}]. But both of them use specific upper bounds \( u_k \) instead of the trivial value \( z^* \).

**Proposition**

\[z_{V}^{\text{LP}}(u) \leq z_{GG}^{\text{LP}}(u) \quad \text{and the inequality holds strict for some vector} \ u \in \mathbb{R}^{|K|} \text{ of upper bounds to the activation level of cutting patterns.} \]

**Proof:** Dantzig-Wolfe decomposition applied to [\text{GG}] transforms it into [\text{VAN}] plus

\[
\sum_{x=1}^{u_k} \lambda_{kx} \leq 1 \quad k \in K \quad (1)
\]

The linear relaxation of [\text{VAN}] + (1) is always equivalent to [\text{GG}^{\text{LP}}] and better than [\text{VAN}^{\text{LP}}] when \( u_k < z^* \).
On lower bounds by linear relaxation

...continue

In fact, the convexification of constraints $x_k \leq u_k \lambda_k$ in [GG] has no effect on $z_{GG}^{LP}(u)$ since the polyhedra

$$P_k = \{(x_k, \lambda_k) \in \mathbb{R}^2 | x_k - u_k \lambda_k \leq 0, x_k \geq 0, 0 \leq \lambda_k \leq 1\} \quad k \in K$$

have integral extreme points $(0,0), (0,1), (u_k,1)$.

On the other hand

$$z_{LP}^{V}(u) = \frac{89}{88} < \frac{228}{225} = z_{GG}^{LP}(u)$$

for $I = \{1,2\}$, $w = 50$, $(w_1, d_1) = (8,182)$, $(w_2, d_2) = (9,91)$, $u_k = \min_{i \in I, a_{ik} > 0} \left\{ z^*, \left\lfloor \frac{d_i}{a_{ik}} \right\rfloor \right\}$

**Remark**

When $u_k = z^*$ inequalities (1) are redundant in [VAN^{LP}]
The whole decomposition scheme

- \([KAN] \supset [KAN^{LP}]\) \rightarrow \([VAN]\)
  - discretization
  - \([VAN^{LP}] \supset [VAN^{LP}] + \Sigma \lambda \leq 1\)

- \([KAN]\)
  - discretization
  - \([KAN^{LP}] \supset [GG^{LP}]\)

- \([GG]\)
  - convexification
  - \([GG^{LP}] \equiv [VAN^{LP}] + \Sigma \lambda \leq 1\)

- \([VAN]\)
  - \([VAN^{LP}] \supset [VAN^{LP}] + \Sigma \lambda \leq 1\)

- \([VAN] + \Sigma \lambda \leq 1\)
Computing the upper bound $u_k$

- Parts cannot be produced more than required
  \[ u_k^V \leq \left\lfloor \frac{d_i}{a_{ik}} \right\rfloor \quad i \in I, a_{ik} > 0 \]
  (Vanderbeck, 1999)

\[ u_k^V = \min_{i \in I, a_{ik} > 0} \left\{ z^*, \left\lfloor \frac{d_i}{a_{ik}} \right\rfloor \right\} \]

- The waste yielded by cutting pattern $k$ cannot be greater than the total waste (Alves, 2005)

\[ u_k^A (w - \sum_{i \in I} w_i a_{ik}) \leq z^* w - \sum_{i \in I} w_i d_i \]

\[ u_k^A = \min \left\{ \frac{z^* w - \sum_{i \in I} w_i d_i}{w - \sum_{i \in I} w_i a_{ik}}, \min_{i \in I, a_{ik} > 0} \left\lfloor \frac{d_i}{a_{ik}} \right\rfloor \right\} \]

- Activating pattern $k$ at level $u_k$ might be infeasible, because the remaining $(z^* - u_k)$ stock items might not be sufficiently many to cover the demand $d'$ not yet fulfilled. This is the case if $\left\lceil z' \right\rceil > z^* - u_k$ where $z'$ is a lower bound to the optimal value of 1-CSP defined on $d'$. In this way a better bound $u_k^B$ can be obtained.
Computational experience

- 375 random instances generated by Cutgen1, with $|I| = 10$, $w = 1000$, and mean part type demand $d = 50$.

The linear relaxation lower bound computed by using $u^B$ is better in 310 cases.

In 43 cases $z^{LP}_{GG}(u^B)$ improves the bound by more than 20%.
We show that reformulation [GG] is in general better than [VAN], and, at least from the theoretical point of view, it should not be discarded.

We observe that the upper bounds on the pattern activation levels play a crucial role in the implementation of a good and practical exact algorithm.

We propose a method to improve the better available upper bound
References


- Vanderbeck, F. 2000. Exact Algorithm for Minimising the Number of Setups in the One-Dimensional Cutting Stock Problem, Operations Research 48 915 926.

2. Reformulations and lower bounds by linear relaxation

Reformulating [Kan] by discretization, see [6], gives tighter bounds to the 1-PMP. Indeed, different master formulations can be drawn from [Kan], depending on the set of dualized constraints. In [5], the author describes a 1-PMP master formulation [Van] obtained by dualizing (1) and (3), or equivalently,

from discretization of the polyedron defined by (2) and (4)-(8).

An alternative master formulation [GG], very close to that of Gilmore and Gomory for the 1-CSP [2] with the addition of fixed setup costs, derives from discretizing the polyedron defined by (2), (6) and (7).
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